Chebychev's Inequality

For a random variable X with mean value $\overline{X}$ and variance σ_X^2, it states that

$$P\left\{\left|X-\overline{X}\right|\geq\varepsilon\right\}\leq\sigma_X^2/\varepsilon^2 \quad \text{for any} \quad \varepsilon>0$$

Proof :

$$\sigma_X^2=\int_{-\infty}^{\infty}(x-\overline{X})^2 f_X(x)dx \geq \int_{\left|x-\overline{X}\right|\geq\varepsilon}(x-\overline{X})^2 f_X(x)dx$$

$$\geq\varepsilon^2\int_{\left|x-\overline{X}\right|\geq\varepsilon} f_X(x)dx=\varepsilon^2 P\left\{\left|X-\overline{X}\right|\geq\varepsilon\right\}$$

Markov's Inequality

For a nonnegative random variable X, and any positive real constant a, we have

$$P\{\mathrm{X} \geq a\} \leq \mathrm{E[X]}/a \qquad a > 0$$

Proof :

$$E[X] = \int_0^{\infty} x f_X(x)dx \geq \int_{x \geq a} x f_X(x)dx \geq \int_{x \geq a} a\, f_X(x)dx$$

$$\geq a \int_{x \geq a} f_X(x)dx = a\, P\{X \geq a\}$$

Chernoff's inequality and Bound

Let X be any random variable, nonnegative or not. For any real $\nu > 0$, it is clear that

$$\exp\left[\nu\left(x - a\right)\right] \geq u(x - a) \qquad (1)$$

where $u(\cdot)$ is the unit-step function and a is an arbitrary real constant.

Since $P\left\{\mathrm{X} \geq a\right\} = \int_{a}^{\infty} f_X(x)dx = \int_{-\infty}^{\infty} f_X(x)u(x-a)dx$, (2) We have

$$P\{X \geq a\} \leq \int_{-\infty}^{\infty} f_X(x) e^{\nu(x-a)} dx = e^{-\nu a} M_X(\nu)$$

where

$$M_X(\nu) = E[e^{\nu X}] \qquad \textit{(moment generating function).}$$

For any random variable X, and any real constant a,

$$P\{X \geq a\} \leq e^{-\nu a} M_X(\nu) \qquad (3)$$

Equation (3) is called *Chernoff's inequality*. Because the right side is a function of parameter ν, it can be minimized with respect to this parameter. The minimum value is called *Chernoff's bound.*