선형대수학 입문

Introduction to Linear Algebra
for engineers and computer scientists

12시간 강좌

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과목 소개

왜 선형대수학을 공부하는가

- **통신공학/신호처리/네트워크이론/제어공학 /채널코딩/암호학** 그리고 **컴퓨터공학** 전분야에서 필수 도구로 사용함.
- 위의 전공 분야는 거의 대부분이 **선형대수학의 기본 이론을 바탕**으로 새로운 연구가 진행됨.
- 예를 들자면,
  - 모든 비선형 방정식은 선형화를 거쳐서 "**선형 연립방정식**"의 형태로 모델링
  - 모든 실계수 미분방정식은 "**선형 연립방정식**"으로 모델링
  - 모든 (통신/압축/제어/랜덤) 신호처리 과정은 "**선형변환**"으로 모델링
  - 관련된 "**행렬**"의 성질을 파악하고 이를 "**변형(대각화)**"하여 처리

본 강좌의 구성

- 총 12시간 수업 + 중간시험+기말시험
  1. Matrices and Simultaneous Linear Equations
  2. Gauss Elimination and LDU Decomposition
  3. Field and Vector Space over a Field
  4. Four fundamental subspaces of an m x n matrix
  5. Matrices and Linear Transformation
  6. Permutation and Determinant
     - 중간시험
  7. Inner Product, Projection, and Least Square Solution
  8. Orthogonal Matrices and QR Decomposition
  9. Orthogonal Subspaces
  10. Eigenvalues and Eigenvectors and Spectral Theorem
  11. Complex matrices, Normal matrices, Similar Transformations and Congruence relation
  12. Positive Definite Matrices and Singular Value Decomposition
      - 기말시험

성적평가 Pass 조건

- **지각 없는 전출**
  - 결석 이후 수업에 참여하지 못함. – nonpass 처리 (이유 없음)
  - 지각하는 경우, 사유서를 수업 이후 당일에 제출. (이메일로)
    - 당일에 제출하지 않으면 nonpass처리함.
  - 1분전에 강의실에 도착할 것.

- 중간시험 100점
- 기말시험 100점
- 총합 140점 이상 획득
  - 중간시험에서 50%의 점수를 획득하지 못한 수강생은 후반기 수업에 참여하지 못함. – nonpass 처리

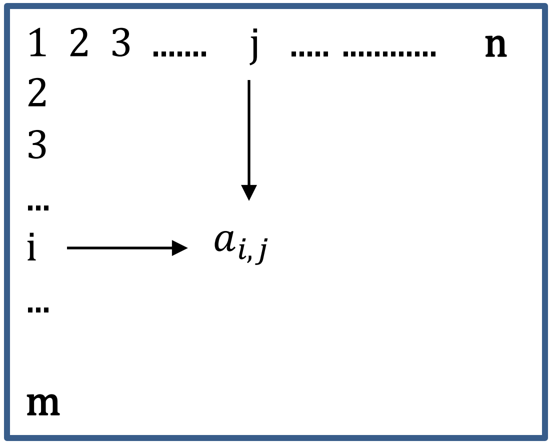
Lecture #1/12

1-1. Matrices
1-2. Simultaneous Linear Equations

1-1. MATRICES

a Matrix of size $m \times n$ over F

components are members of F



$= A = (a_{i,j})$

- $\boldsymbol{R}$ **– real numbers**
- Q – rational numbers
- C – complex numbers
- Z – integers (not field)
- Z_n - integers mod n
- $\boldsymbol{Z_p}$ **- integers mod $\boldsymbol{p}$**
- $\boldsymbol{F_2}$ **- binary numbers**
- etc...

$a_{i,j}$ $= (i,j)$-component of A

(i,j)원소, (i,j)성분

column (열) vector

row (행) vector

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Matrix addition

- Let A and B be two matrices over the same F
- A+B is possible if both A and B have the same size.
- A+B has the same size as A and B.

- We say

addition of matrices A and B

$$A+B=C=(c_{ij})$$

$$\text{where } c_{ij}=a_{ij}+b_{ij} \text{ for all } i \text{ and } j$$

addition defined on F

Matrix multiplication

- Let A and B be two matrices over the same F
- The multiplication AB (or A·B) is possible if the number of columns of A is the same as the number of rows of B.

multiplication of matrices A and B

- We say

multiplication defined on F

$$AB = C = (c_{ij})$$

$$\text{where } c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj}$$

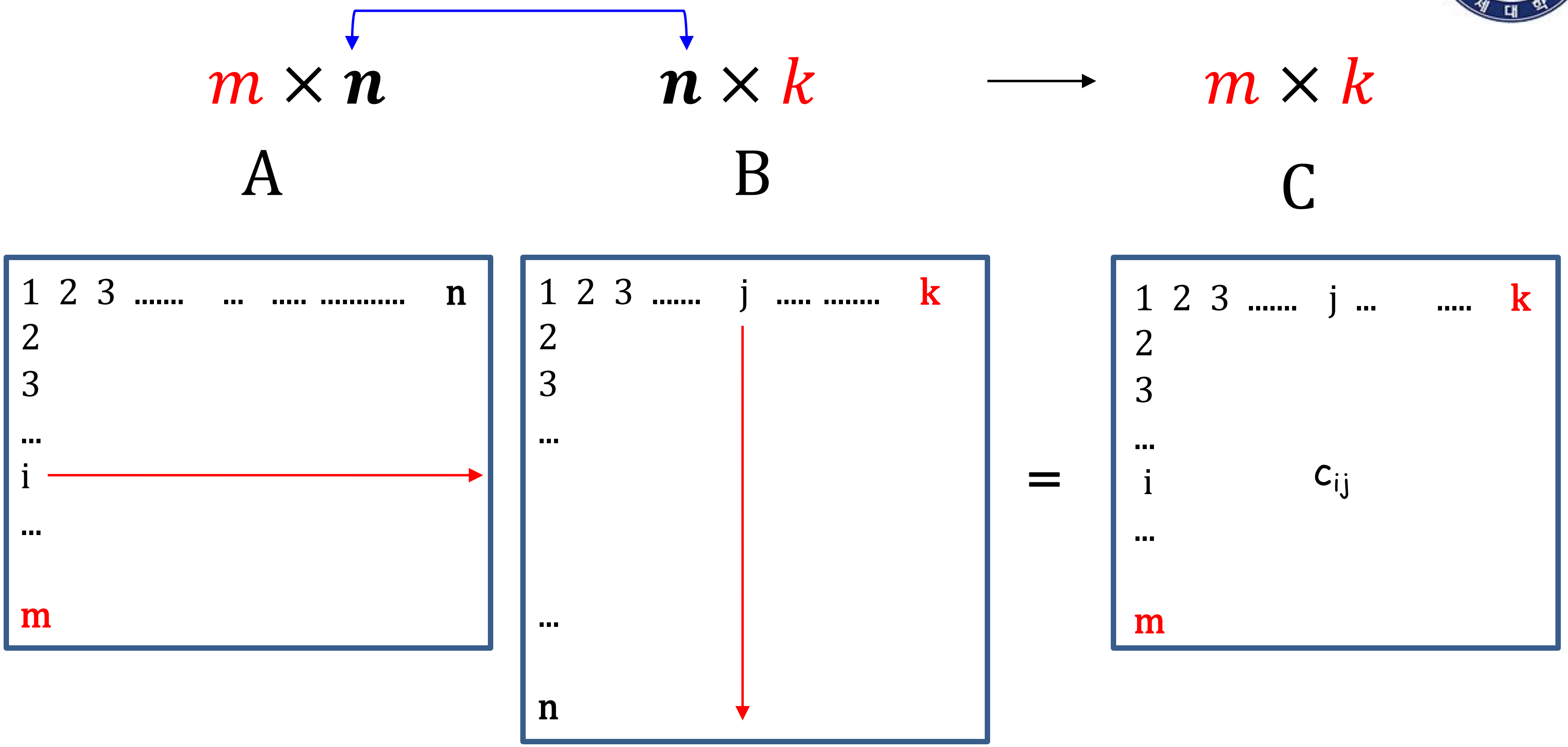
or

addition defined on F

$$c_{ij} = \sum_{l=1}^{n} a_{il} b_{lj}$$

when n = number of **columns** of A

= number of **rows** of B

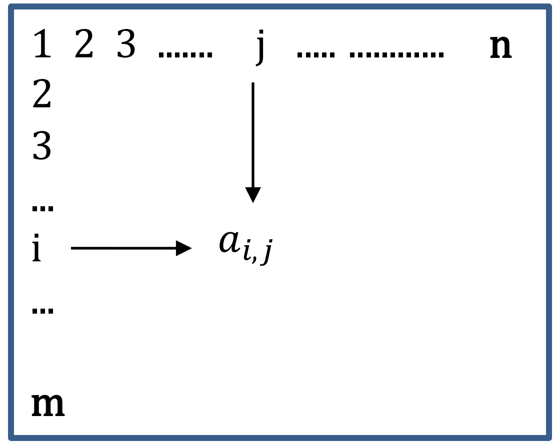


$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj}$$

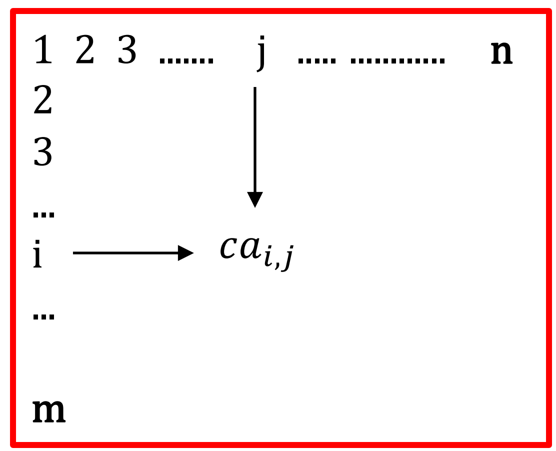
= "dot product" of i-th row of A and j-th column of B

Multiplication by a constant

$$A = (a_{i,j}) =$$



$$cA = c(a_{i,j}) = (ca_{i,j}) =$$



Some Properties

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

- Zero matrix: all the components are 0.

- Identity matrix: all the components are 0, except for the diagonal components which are all 1. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
  - It is a **square** matrix.

- A+B=B+A for **any** two same size matrices
- AB≠BA in general

- A+B+C=A+B+C=A+B+C if A+B, B+C are possible.

  ⇔ A,B,C all have the same size
- A(BC)=ABC=(AB)C if AB, BC are possible.
- A(B+C)=AB+AC

연습문제 1-1

다음 주어진 행렬들에 대해서 아래 각 문항의 행렬을 계산하여라.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \end{pmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 3 & -1 \end{bmatrix}, \quad C = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$

(a) AB (b) BA (c) $(AB)C$

(d) $A(BC)$ (e) CD (f) DC

1-2. SIMULTANEOUS LINEAR EQUATIONS

Simultaneous Linear Equations
2 x 2 example

Consider

$$2x - y = 1 \quad \text{-------} \quad ①$$

$$x + y = 5 \quad \text{-------} \quad ②$$

Row view
Column view

$$+ \quad \underline{}$$

$$3x \quad = 6$$

or x=2

substitute back to ②: y=3

yes.
x=2 and y=3
is a solution

Two views of S.L.E.

- Row view
  - Rows are lines
  - Solutions (if any) are the intersections of these lines(rows)

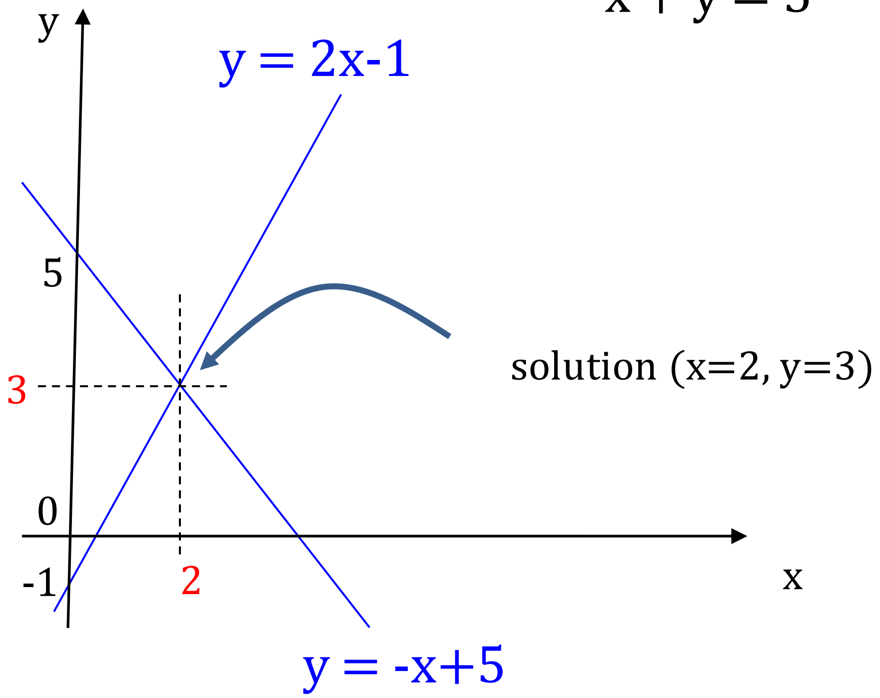
- Column view
  - Columns are weighted and summed
  - Solutions (if any) are the appropriate weights of the columns

Row View

- Rows are lines
- Solutions (if any) are the intersections of these lines(rows)

$2x - y = 1 \longleftrightarrow y = 2x-1$

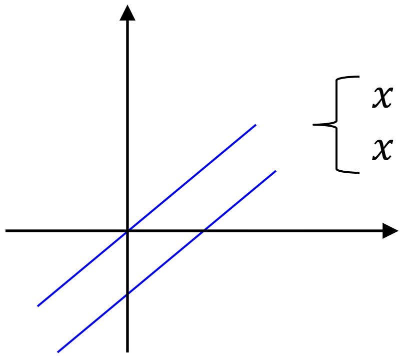
$x + y = 5 \longleftrightarrow y = -x+5$



THEOREM 1.

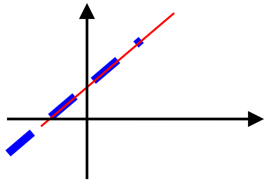
in general,

(1) no solution
when they are parallel



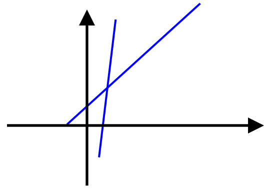
$$\begin{cases} x - y = 0 \\ x - y = 1 \end{cases}$$

(2) many solutions
when they coincide



$$\begin{cases} x - y = 1 \\ 2x - 2y = 2 \end{cases}$$

(3) unique solution
in all other cases



Column View

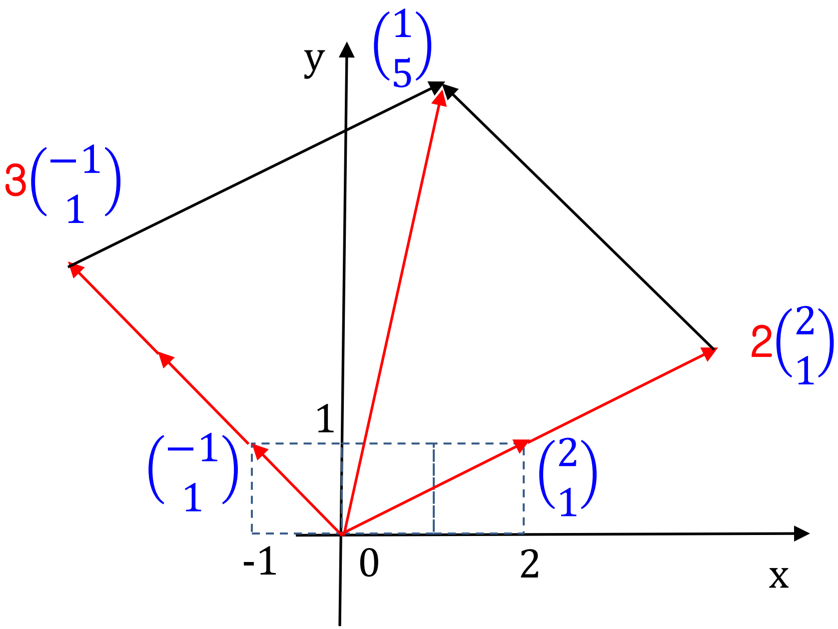
- Columns are weighted and summed
- Solutions (if any) are the appropriate weights of the columns

$$
\begin{aligned}
2x \quad - y \quad &= \quad 1 \\
x \quad + y \quad &= \quad 5
\end{aligned}
$$

$$\Leftrightarrow \quad x\begin{pmatrix}2\\1\end{pmatrix} + y\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}1\\5\end{pmatrix}$$

column vectors

$ax + by = c$



$$2\begin{pmatrix}2\\1\end{pmatrix} + 3\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}1\\5\end{pmatrix}$$

solution !!

column vectors

$a\color{red}{x} + b\color{red}{y} = c$

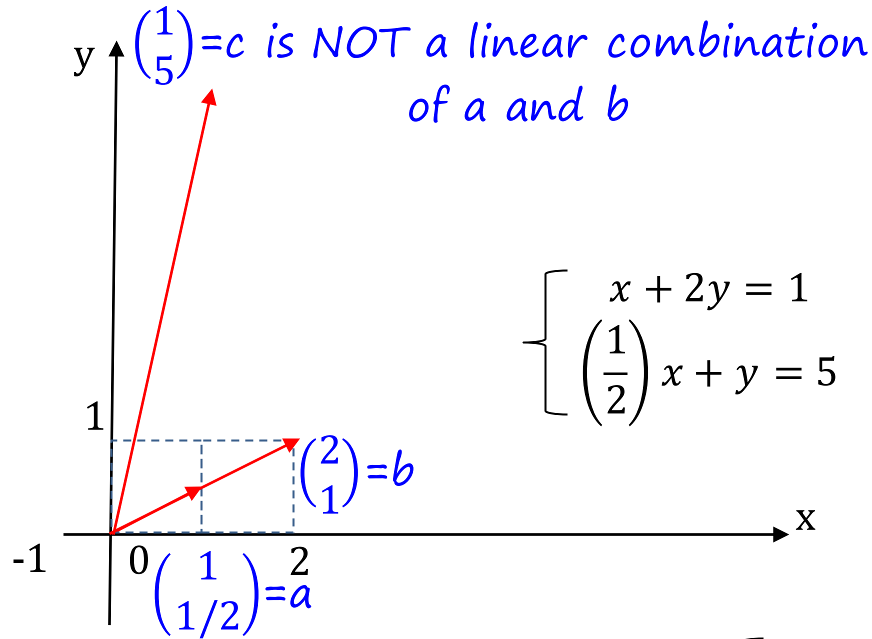
THEOREM 2.

- no solution if c is not a **linear combination** of a and b

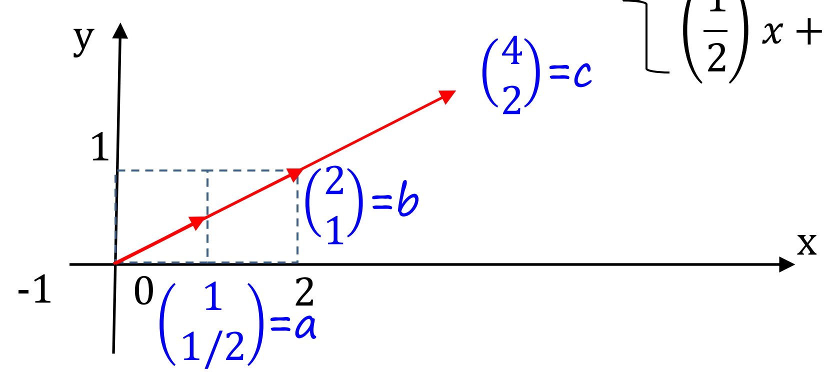
  $\leftrightarrow$ a and b are on the same line, but c is **not** on this line

- at least one solution if c belongs to **SPAN** of a and b

  many solution
  $\leftrightarrow$ a, b and c are on the same line
  unique solution
  $\leftrightarrow$ otherwise



$$\begin{cases} x + 2y = 1 \\ \left(\frac{1}{2}\right) x + y = 5 \end{cases}$$



$$\begin{cases} x + 2y = 4 \\ \left(\frac{1}{2}\right) x + y = 2 \end{cases}$$

linear combination of vectors

- $\begin{pmatrix}2\\1\end{pmatrix}$ 와 $\begin{pmatrix}-1\\1\end{pmatrix}$ 의 linear combination은 $x\begin{pmatrix}2\\1\end{pmatrix}+y\begin{pmatrix}-1\\1\end{pmatrix}$ 형태의 모든 가능한 vector를 뜻함.
  - 여기서 x와 y는 임의의 실수.

- Question: $\begin{pmatrix}2\\1\end{pmatrix}$ 와 $\begin{pmatrix}-1\\1\end{pmatrix}$ 의 linear combination으로 임의의 벡터 $\begin{pmatrix}c\\d\end{pmatrix}$를 만들 수 있는가?
  - $\begin{pmatrix}2\\1\end{pmatrix}$ 와 $\begin{pmatrix}-1\\1\end{pmatrix}$ 의 관계에 따라, 가능할 수도. 불가능할 수도 있음.
  - 이 경우에는 가능함. Why??
  - $\begin{pmatrix}1\\0\end{pmatrix}$ 와 $\begin{pmatrix}0\\1\end{pmatrix}$의 경우에는??
  - $\begin{pmatrix}1\\1\end{pmatrix}$ 와 $\begin{pmatrix}2\\2\end{pmatrix}$의 경우에는??

$$\begin{pmatrix}c\\d\end{pmatrix}=c\begin{pmatrix}1\\0\end{pmatrix}+d\begin{pmatrix}0\\1\end{pmatrix}$$

- 3개 혹은 그 이상의 벡터에 대해서도 마찬가지.
- 2-tuple이 아니라 3-tuple, 4-tuple, 등등에서도 마찬가지.

span

- Given a set S={v_1,v_2,...} of vectors, their span is the set of **all the linear combinations** of the vectors in S={v_1,v_2,...}.

- span(S) = { $\sum_i a_i v_i$ | $a_i \in R$}

- Example: S={ $\binom{2}{1}$, $\binom{-1}{1}$ } → span(S) = {x-y plane 전체}

- Example: S={ $\binom{1}{1}$, $\binom{2}{2}$ } → span(S) = {the line y=x}

연습문제 1-2

다음 선형연립방정식의 해의 존재 유무를 row-view와 column-view 관점에서 설명하고, 유일 해를 가지면 유일 해를 구하고, 무수히 많은 해를 가지면 서로 다른 2개의 해를 구하여라.

(1)

$$\begin{cases} 2x + y = 1 \\ 4x + 2y = 0 \end{cases}$$

(2)

$$\begin{cases} 2x + y = 1 \\ 4x + 2y = 2 \end{cases}$$

(3)

$$\begin{cases} 2x + y = 1 \\ x + 2y = 0 \end{cases}$$