Polyphase Sequences with Almost Perfect Autocorrelation and Optimal Crosscorrelation

2020 KICS-NA Workshop

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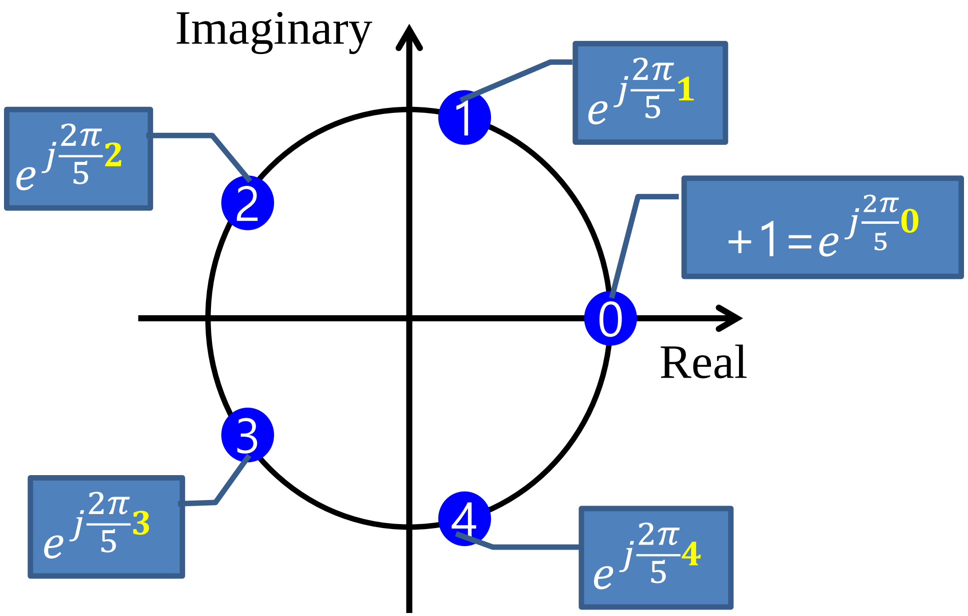
- Introduction
- Some historical reviews on
  - the design of polyphase sequences family with GOOD (complex) correlation properties based on **Sidelnikov sequences** of period $q-1$
  - the design of polyphase sequences family with GOOD (complex) correlation properties based on **Partial Residue sequences** of period p (**another type of Sidelnikov sequences**)
- Recent development on **Partial Residue sequences**
- Main Result on **Sidelnikov sequences**
- Some discussion and Conclusion

Polyphase sequences

Alphabet of a polyphase sequence



$\{x(n)\}_{n=0}^{L-1}$ be ***k*-ary polyphase sequences** of length L

$\Updownarrow$

$x(n)$ belongs to the integers mod ***k*** for each $n = 0,1,\ldots$

Equivalent representations

A complex-valued polyphase sequence

$$e^{j\frac{2\pi}{5}1},\ e^{j\frac{2\pi}{5}3},\ e^{j\frac{2\pi}{5}0},\ e^{j\frac{2\pi}{5}2},\ e^{j\frac{2\pi}{5}4},\ldots$$

$\Updownarrow$

1, 3, 0, 2, 4, …

Corresponding **phase sequence** over the integers **modulo 5**

Correlation of sequences

- Let $\boldsymbol{x} = \{x(n)\}_{n=0}^{L-1}$ and $\boldsymbol{y} = \{y(n)\}_{n=0}^{L-1}$ be two ***k*-ary polyphase sequences** of length L. (over the integers mod $\boldsymbol{k}$)
- The (periodic) correlation between $\boldsymbol{x}$ and $\boldsymbol{y}$ at time shift τ is computed over the complex:

$$\mathrm{C}_{\boldsymbol{x},\boldsymbol{y}}(\tau) = \sum_{n=0}^{L-1} \omega^{x(n)} \left(\omega^{y(n+\tau)}\right)^* = \sum_{n=0}^{L-1} \omega^{x(n)-y(n+\tau)}$$

where $\omega = e^{-j\frac{2\pi}{k}}$ is a complex primitive k-th root of unity.

  - It is called autocorrelation if $\boldsymbol{y} = \boldsymbol{x}$.
  - It is called cross-correlation **otherwise.**

In the beginning

- (Sidelnikov-69) Sidelnikov introduced two different types of non-binary (k-ary polyphase) sequences with

  very good non-trivial autocorrelation (ONLY)

  - **Power Residue sequences** (PRS in short) **of period p**
  - **Sidelnikov sequences of period $q-1$**

  ✓ V. M. Sidelnikov, "Some k-valued pseudo-random sequences and nearly equidistant codes," *Probl. Inf. Transm.*, vol. 5, pp. 12-16, 1969.

- (Lempel-Cohn-Eastman-77) re-discovered binary "Sidelnikov sequences" of period $q-1$

  ✓ A. Lempel, M. Cohn, and W.L. Eastman, "A class of binary sequences with optimal **autocorrelation** properties," *IEEE* Trans. *Inform. Theory,* vol. *23,* No. 1, pp. 38-42, Jan. 1977.

Cosets of k-th powers in F_q^*

- p = odd prime, $q = p^m$ and F_q = finite field of size q
- μ = primitive element of F_q
- k is a divisor of $q-1$ so that $q = kf + 1$ for some f
- Coset Partition
  - ✓ D_0 = set of k-th powers in F_q^*

    $$= \{\mu^{k0} = 1,\ \mu^{2k},\ \mu^{3k},\ \dots,\ \mu^{(f-1)k}\}$$
  - ✓ $D_i = \mu^i D_0$ for $i = 0,1,\dots,k-1$

    $$= \{\mu^{k0+i} = \mu^i, \mu^{2k+i}, \mu^{3k+i}, \dots, \mu^{(f-1)k+i}\}$$
- Well-known that

$$F_q^* = \bigcup_{i=0}^{k-1} D_i \quad \text{is a disjoint union}$$

and

$$|D_i| = f \ \text{ for all } i = 0,1,\dots,k-1.$$

Example

- Let $q = 13$ and the finite field $F_q = F_{13}$ has $\mu = 2$ *(primitive)* since

$$\{\mu^n | n = 1,2,\dots,11,12\}$$
$$= \{\mu^1, \mu^2, \mu^3, \mu^4, \dots, \mu^{12}\}$$
$$= \{2,4,8,3,6,12,11,9,5,10,7,1\} = F_{13}^*$$

- A divisor $k = 3$ of $q - 1 = 12 = 3 \times 4$ with $f = 4 = (q-1)/k$

and $$D_0 = \{2^3, 2^{3\cdot 2}, 2^{3\cdot 3}, 2^{3\cdot 4}\} = \{8,\ 12,\ 5,\ 1\}$$

is the set of all the k-th (3rd) powers of F_{13}^*.

- All its cosets are

$$D_0 = 2^0 D_0 = \{\mathbf{8, 12,\ 5,\ 1}\}$$
$$D_1 = 2^1 D_0 = \{\mathbf{3, 11, 10,\ 2}\}$$
$$D_2 = 2^2 D_0 = \{\mathbf{6,\ 9,\ 7,\ 4}\}$$

each of size $f = 4$, and

$F_{13}^* = D_0 \cup D_1 \cup D_2$ is a disjoint union

Two sequences from Sidelnikov

- Let p must be an **odd prime** and $q = p^m$
  - Let $k \geq 2$ be a **divisor** of $q-1$
  - Let μ be a primitive element of F_q^*
  - D_0 = set of all the k-th powers of F_q^* (for Sidel S)
  - $D_i = \mu^i D_0$ = coset of D_0 for $i = 0,1,\ldots,k-1$

- A k-ary power residue sequence (PRS) of **period $q = p$**

  ($q = p$ =prime):

$$s(n) = \begin{cases} 0, & \textbf{if} \quad n = 0 \\ i, & \textbf{if} \quad n \in D_i \end{cases}$$

- A k-ary sidelnikov sequence of **period $q-1$**

  ($q - 1 = p^m - 1$ = one less than a prime or a power of a prime)

$$s(n) = \begin{cases} 0, & \text{if} \quad \mu^n + 1 = 0 \\ i, & \text{if} \quad \mu^n + 1 \in D_i \end{cases}$$

$p = q = 13$ *and* $\boldsymbol{k} = \mathbf{3}$

- $D_0 = 2^0 D_0 = \{8, 12, 5, 1\}$
- $D_1 = 2^1 D_0 = \{3, 11, 10, 2\}$
- $D_2 = 2^2 D_0 = \{6, 9, 7, 4\}$

- A k-ary PR S of period p:

$$s(n) = \begin{cases} 0, & \textbf{if} \quad n = 0 \\ i, & \textbf{if} \quad n \in D_i \end{cases}$$

- A k-ary Sidel. sequence of period $q-1$:

$$s(n) = \begin{cases} 0, & \text{if} \quad \mu^n + 1 = 0 \\ i, & \text{if} \quad \mu^n + 1 \in D_i \end{cases}$$

$\boldsymbol{n}$	0	1	2	3	4	5	6	7	8	9	10	11	12
PRS	0	0	1	1	2	0	2	2	0	2	1	1	0
μ^n	1	2	4	8	3	6	12	11	9	5	10	7	
μ^n+1	2	3	5	9	4	7	0	12	10	6	11	8	
Sidel S	1	1	0	2	2	2	0	0	1	2	1	0	X
$\log_\mu (\mu^n+1)$	1	4	9	8	2	11	*	11	10	5	7	3	X
$\log_\mu (\mu^n+1)$ **mod 3**	1	1	0	2	2	2	0	0	1	2	1	0	X

equivalent presentation: $s(n) = \log_\mu (\mu^n+1) \bmod \boldsymbol{k}$ - **GONG-10**

QUESTIONs

Can we construct a set of sequences with

GOOD cross-correlation

as well as

GOOD non-trivial autocorrelation

from any of these sequences?

Up until 2006, **only** the **autocorrelation** properties of these sequences are known (original paper Sidelnikov-69):

The non-trivial autocorrelation magnitude is upper bounded by 3 (for PRS) or 4 (for Sidel. sequences).

First Attempt (2006-2007)

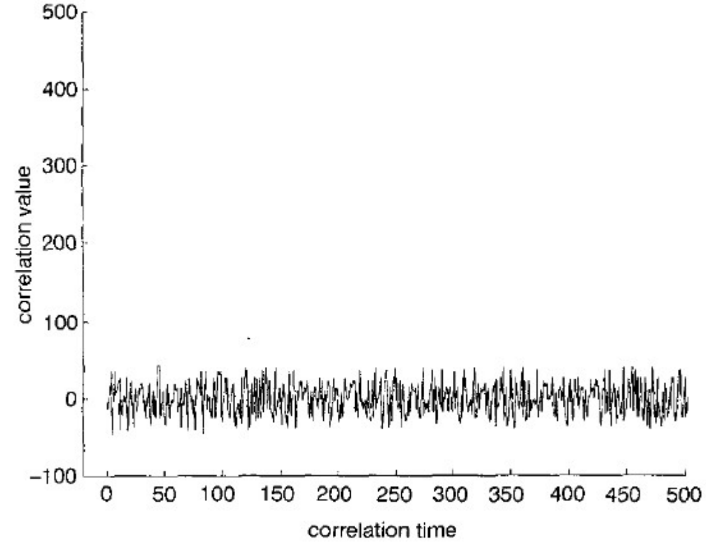
- Construct a family from a given sequence by changing the primitive element in the definition.
- It turned out that the same family can be obtained by multiplying a constant term-by-term.
- Results are
  - PRS (period p): **Song-06** (ISIT)
    - $Max \leq \sqrt{p} + 2$

      Crosscorrelation of q-ary power residue sequences of period p
  - SS (period q-1): **Song-07** (IT Trans.)
    - $Max \leq \sqrt{q} + 3$

      Crosscorrelation of Sidel'nikov Sequences and Their Constant Multiples
- Note that the size of the family is $k-1$ for k-ary sequences. It is only $\varphi(k)$ when we need to maintain k distinct values.

An improvement begins by some observations and a conjecture

- Z. Guohua and Z. Quan, "Pseudonoise codes constructed by Legendre sequence," IEE Electronic Letters, vol. 38, no. 8, pp. 376-377, 2002.
- The technique of shift-and-add (as in the construction of GOLD sequences using an m-sequence) is introduced.
- They used a Legendre sequence and the technique of shift-and-add to construct a family with good crosscorrelation, where the crosscorrelation is (conjectured to be) upper bounded by $4\lfloor 2\sqrt{p}/4 \rfloor + 1$



It is proved by Rushanan at ISIT-06

- J. Rushanan, "Weil Sequences: A Family of Binary Sequences with Good orrelation Properties," *Proc. of IEEE Int. Symp. Information Theory(ISIT2006)*, Seattle, WA, USA, July 2006.

- Crosscorrelation of the sequence family containing a Legendre sequence and its shift-and-add sequences is upper bounded by $\mathbf{2\sqrt{p}+5}$.

- Major Technique:

quartic polynomial (product of 4 linear polynomials)

$$\left|\sum_{x=0}^{p-1}\left(\frac{(x+a_1)\cdots(x+a_4)}{p}\right)\right| \le 2\sqrt{p}+1$$

quadratic character

Results of No-Chung/Yang/Gong (2008-2010)

Shift-and-add techniques

to construct larger family of sequences from a Sidelnikov sequence or a power-residue sequence

➕

Weil Bound on character sums

to prove crosscorrelation bound of the family constructed

Sidelnikov sequences only

- **Y.-S. Kim, J.-S. Chung, J.-S. No, and H. Chung**, "New families of M-ary sequences with low correlation constructed from **Sidel'nikov sequences**," *IEEE Trans. Inf. Theory*, vol. 54, no. 8, pp. 3768–3774, Aug. **2008**.

Both Sidelnikov sequences and PRS

- **Y. K. Han and K. Yang**, New **M-ary sequence families** with low correlation and large size, *IEEE Trans. Inf. Theory*, vol. 55, no. 4, pp. 1815-1823, Apr. **2009**.
- **N. Y. Yu and G. Gong**, Multiplicative Characters, the Weil Bound, and **Polyphase Sequence Families** With Low Correlation, *IEEE Trans. Inf. Theory*, vol. 56, no. 12, pp. 6376-6387, Dec. **2010**.

Note that the size of the family becomes $\approx \boldsymbol{kq/2}$ for $\boldsymbol{k}$-ary sequences of period $\boldsymbol{q-1}$.

Array structure of Sidelnikov sequences

For a k-ary **Sidelnikov sequence** $s(t)$ of period $\boldsymbol{q^d - 1}$, make an array as

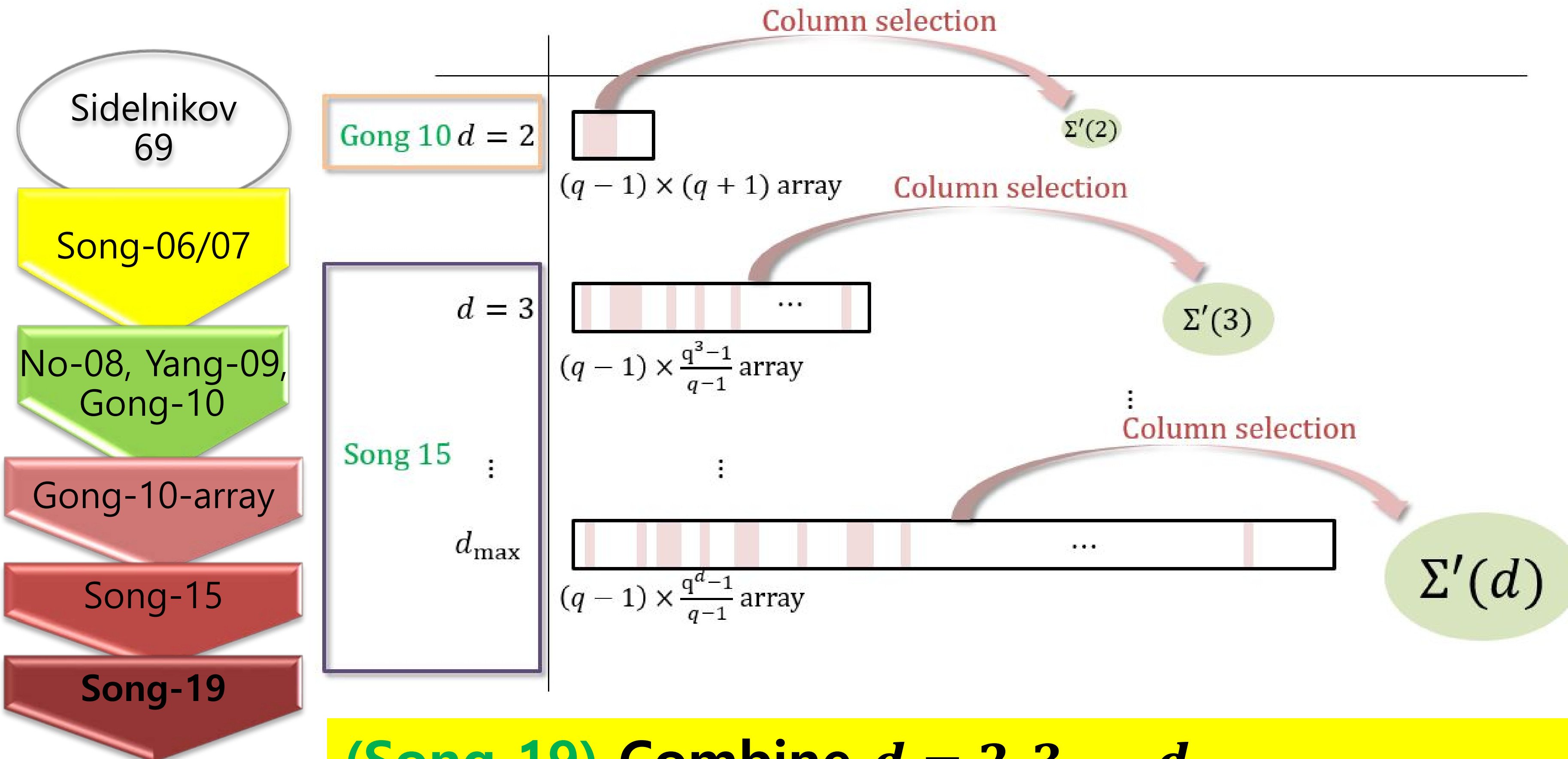
$$\begin{pmatrix} s(0) & s(1) & \cdots & s(\frac{q^d-1}{q-1}-1) \\ s(\frac{q^d-1}{q-1}) & s(\frac{q^d-1}{q-1}+1) & \cdots & s(2\times\frac{q^d-1}{q-1}-1) \\ \vdots & \vdots & \ddots & \vdots \\ s((q-2)\times\frac{q^d-1}{q-1}) & s((q-2)\times\frac{q^d-1}{q-1}+1) & \cdots & s(q^d-2) \end{pmatrix}$$

and **choose some columns** to construct a set of k-ary sequences of period $\boldsymbol{q-1}$.

(Gong 10) when $\boldsymbol{d = 2}$
(Song 15) when $\boldsymbol{3 \le d < \sqrt{q}/2}$ with $q \geq 27$

The **family size** now becomes $\approx \boldsymbol{kq^d/d}$

Array structure of **Sidelnikov sequences**



(Song 19) Combine $d = 2, 3, \dots, d_{max}$,

The **family size** now becomes $\approx k \sum q^d / d$

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