직교라틴방진으로부터 마방진 만들기

- 최석정의 9차 직교라틴방진을 중심으로

최석정과 현대수학
대한수학회 최석정상 제정 기념 학술대회

2021. 12. 14

송홍엽
연세대학교 전기전자공학부

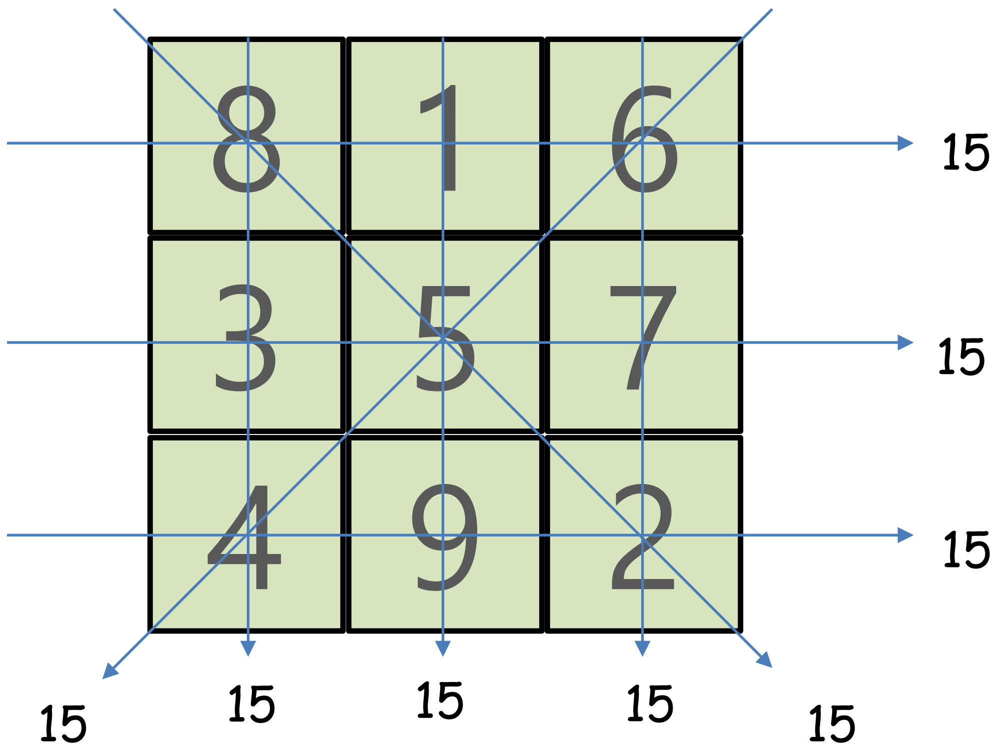
목차

여기가 제일 어렵습니다.
이후부턴 매우 쉽습니다.^^

논문준비중입니다.
매우 쉽습니다.^^

(1) 서론 – 역사적 사실과 기본적 용어정리

3차 마방진 (Magic Square)



3차 마방진: 1부터 9까지의 숫자에서 3개를 선택하여 합이 15가 되는 8가지 **모든 경우**를 사용함.

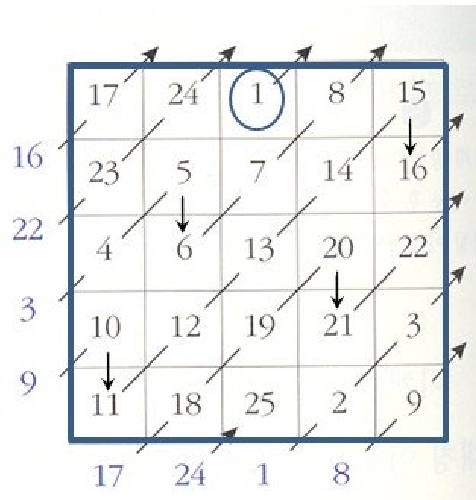
4차 부터는 이러한 특성은 만족하지 않는다.

왜냐면 1부터 16까지 숫자에서 4개를 선택하여 합이 34가 되는 경우는 최소 4+4+2+1 가지 있다.

n차 마방진 = 1부터 n^2까지의 숫자를 한번씩 사용한 정방진으로 각 **행**과 **열** 그리고 **대각선**의 합이 같다.

준마방진 (semi-magic) = 각 행과 열의 합이 같다.

Order 5



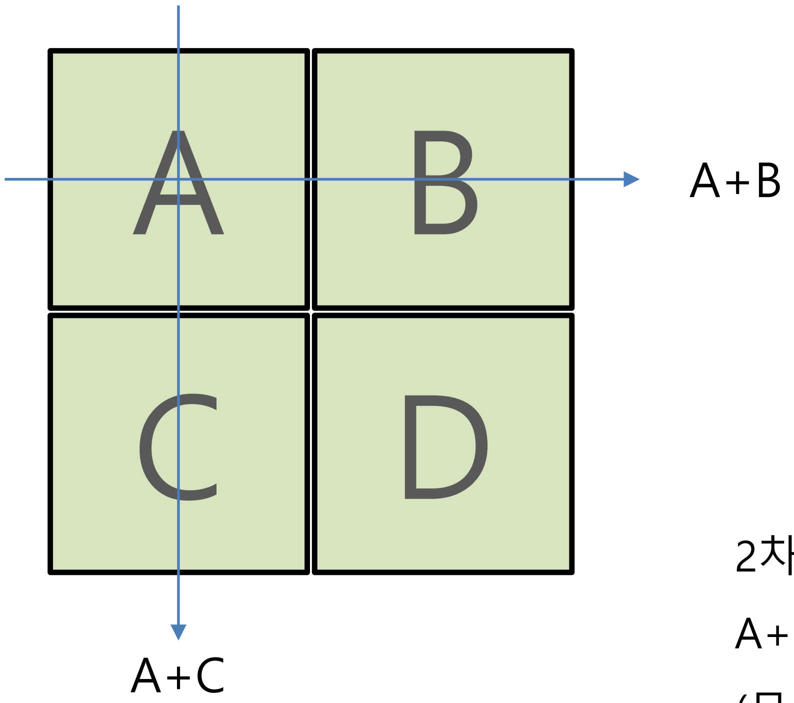
Order 9

47	58	69	80	1	12	23	34	45
57	68	79	9	11	22	33	44	46
67	78	8	10	21	32	43	54	56
77	7	18	20	31	42	53	55	66
6	17	19	30	41	52	63	65	76
16	27	29	40	51	62	64	75	5
26	28	39	50	61	72	74	4	15
36	38	49	60	71	73	3	14	25
37	48	59	70	81	2	13	24	35

홀수차 마방진 생성법

수학적 증명? will come very soon…

2차 마방진은 존재하지 않는다.



2차 마방진이라면

A+B=A+C 이므로 B=C 이다.

(모순)

Order 4

1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

?

14	9	3	8
7	4	10	13
12	15	5	2
1	6	16	11

1	2	33	34	35	6
30	8	28	27	11	7
19	23	15	16	14	24
18	20	21	22	17	13
12	26	10	9	29	25
31	32	4	3	5	36

n차 라틴방진

1차

1

2차

1	2
2	1

3차

1	2	3
3	1	2
2	3	1

4차

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

......

n개의 서로다른 기호(숫자, 문자)를 각행과 열에 한번씩 사용하며

각각의 **행**과 **열**에 중복이 없어야 한다.

직교라틴방진 = Pair of Orthogonal Latin squares (**POLS**)

= 그레코-라틴방진 = Graeco-Latin squares

= **오일러**방진 = **Euler** squares

α	β	γ
γ	α	β
β	γ	α

a	**b**	**c**
b	**c**	**a**
c	**a**	**b**

3차 직교라틴방진(쌍)

⟺

두개의 라틴방진을 겹쳐서 얻어지는 모든 "순서쌍"이 다르다

α**a**	β**b**	γ**c**
γ**b**	α**c**	β**a**
β**c**	γ**a**	α**b**

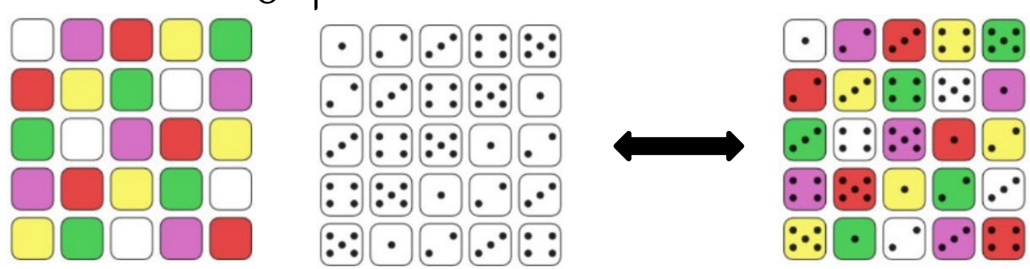
3차 직교라틴방진

4차

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

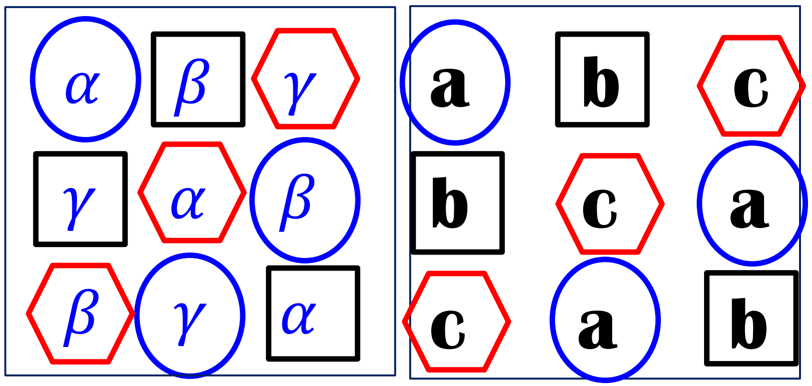
1	2	3	4
3	4	1	2
4	3	2	1
2	1	4	3

5차



직교라틴방진

A latin square has an **orthogonal mate** if and only if it can be decomposed of n **disjoint transversals**.



αa	βb	γc
γb	αc	βa
βc	γa	αb

A **transversal** in a latin square is a set of n positions, one from each row and column, containing each of n symbols exactly once.

직교라틴방진

6차는 못찾겠다.　　혹시,　없는가?

… 존재 안하는가?

7차..
8차..
9차..

10차도 없는가???
…

(1782) Euler Conjecture:

No POLS of order 4k+2

(1900) - no 6차
(1959) - yes 10차
(1960) - POLS of all orders 4k+2 except 6

마방진을

만들기 위하여

직교라틴방진을

생성함

Leonhard Euler (1707년 - 1783년)
스위스 출생.
러시아와 독일에서 활동.

1782 논문발표

1776 구두발표

마방진을

만들기 위하여

직교라틴방진을

생성함

Leonhard Euler (1707년 - 1783년)
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러시아와 독일에서 활동.

L. Euler의

마방진(그리고 **직교라틴방진**)에 관한 논문

1782 논문발표

1776 구두발표

(
Combinatorial Mathematics
Combinatorial Design
조합수학
)

최초 시작으로
수학사는 기록함

최석정(崔錫鼎) (1646년~1715년)
영의정: 1701-1710
《구수략》 저술. 1710~1715(??)

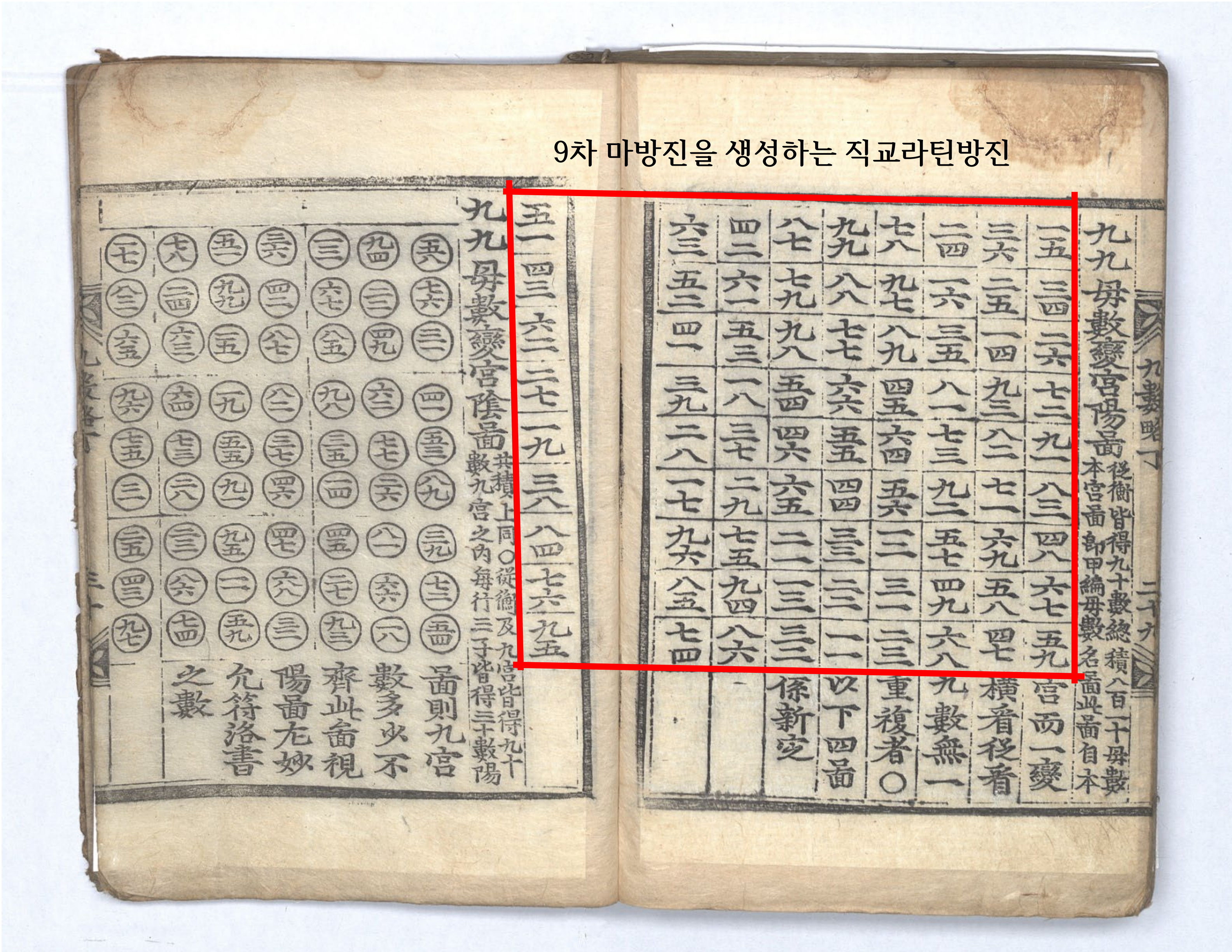
마방진을
만들기 위하여
직교라틴방진을
생성함

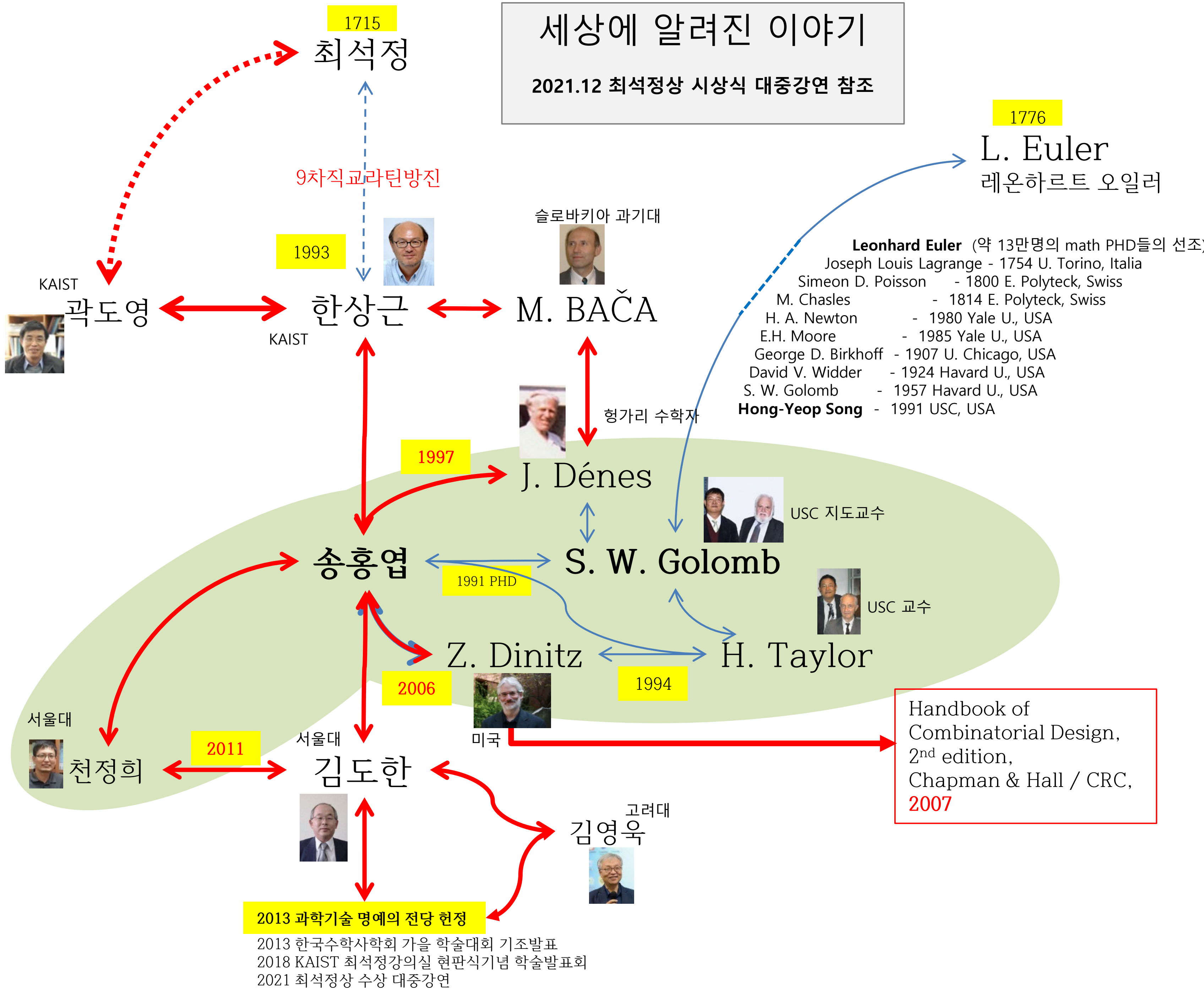
Leonhard Euler (1707년 - 1783년)
스위스 출생.
러시아와 독일에서 활동.

1715(?)
67 년(?) → 1782 논문발표
61 년(?) → 1776 구두발표

그보다 약 60여년 전

(**목판본**) - 연세대학교 학술정보원 고문서 보관실




세상에 알려진 이야기
2021.12 최석정상 시상식 대중강연 참조
1715
최석정
9차직교라틴방진
1776
L. Euler
레온하르트 오일러
Leonhard Euler (약 13만명의 math PHD들의 선조)
Joseph Louis Lagrange - 1754 U. Torino, Italia
Simeon D. Poisson - 1800 E. Polyteck, Swiss
M. Chasles - 1814 E. Polyteck, Swiss
H. A. Newton - 1980 Yale U., USA
E.H. Moore - 1985 Yale U., USA
George D. Birkhoff - 1907 U. Chicago, USA
David V. Widder - 1924 Havard U., USA
S. W. Golomb - 1957 Havard U., USA
Hong-Yeop Song - 1991 USC, USA
KAIST
곽도영
1993
한상근
KAIST
슬로바키아 과기대
M. BAČA
헝가리 수학자
1997
J. Dénes
USC 지도교수
송홍엽
S. W. Golomb
1991 PHD
USC 교수
Z. Dinitz
H. Taylor
2006
1994
미국
Handbook of
Combinatorial Design,
2nd edition,
Chapman & Hall / CRC,
2007
서울대
천정희
2011
서울대
김도한
고려대
김영욱
2013 과학기술 명예의 전당 헌정
2013 한국수학사학회 가을 학술대회 기조발표
2018 KAIST 최석정강의실 현판식기념 학술발표회
2021 최석정상 수상 대중강연

(2) 최적정 직교라틴방진의 특성

- 마방진을 생성한다
- Palindrom 특성
- Kim&Prasanna의 직교라틴방진과 사실상 같다
- Local 구조를 가진다

마방진을 생성한다

월간 과학동아, 2008년 8월호

최석정이 만든 9차 그레코-라틴방진 by canonical map

'구수략'에 소개된 9차 그레코-라틴방진(왼쪽). 각 성분의 첫째 수 p에서 1을 빼고 9를 곱한 뒤 둘째 수 q를 더한 값을 가진 배열을 만들면 9차 마방진(오른쪽)이 된다.

5 1	6 3	4 2	8 7	9 9	7 8	2 4	3 6	1 5
4 3	5 2	6 1	7 9	8 8	9 7	1 6	2 5	3 4
6 2	4 1	5 3	9 8	7 7	8 9	3 5	1 4	2 6
2 7	3 9	1 8	5 4	6 6	4 5	8 1	9 3	7 2
1 9	2 8	3 7	4 6	5 5	6 4	7 3	8 2	9 1
3 8	1 7	2 9	6 5	4 4	5 6	9 2	7 1	8 3
8 4	9 6	7 5	2 1	3 3	1 2	5 7	6 9	4 8
7 6	8 5	9 4	1 3	2 2	3 1	4 9	5 8	6 7
9 5	7 4	8 6	3 2	1 1	2 3	6 8	4 7	5 9

$9(p-1)+q$ →

37	48	29	70	81	62	13	24	5
30	38	46	63	71	79	6	14	22
47	28	39	80	61	72	23	4	15
16	27	8	40	51	32	64	75	56
9	17	25	33	41	49	57	65	73
26	7	18	50	31	42	74	55	66
67	78	59	10	21	2	43	54	35
60	68	76	3	11	19	36	44	52
77	58	69	20	1	12	53	34	45

n=9 and two of the four diagonals of the squares have the constant value 5

최석정의 9차 직교라틴방진

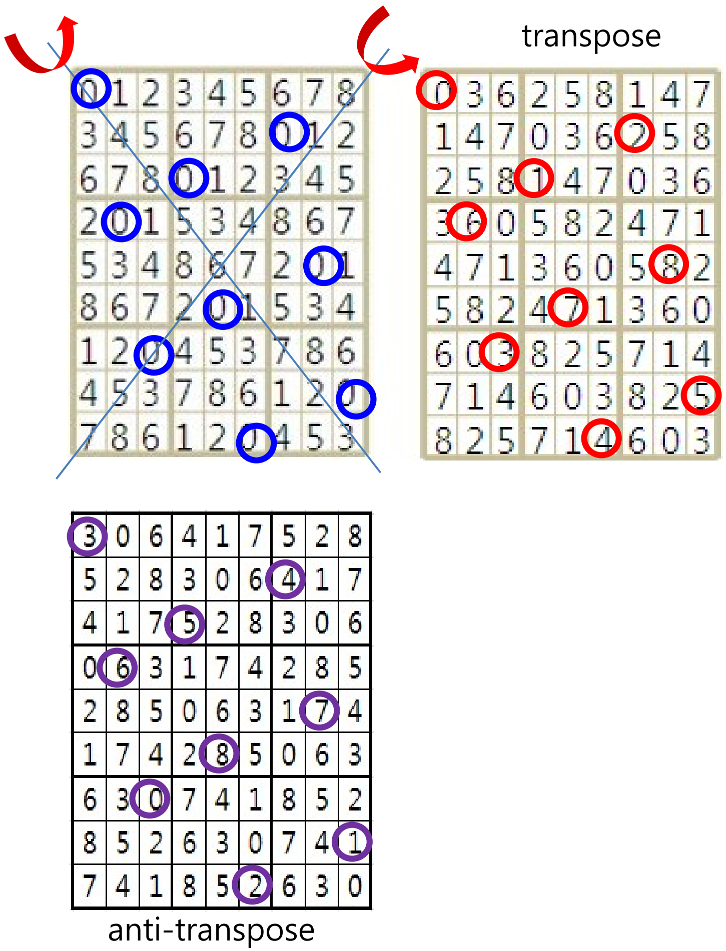
5	6	4	8	9	7	2	3	1
4	5	6	7	8	9	1	2	3
6	4	5	9	7	8	3	1	2
2	3	1	5	6	4	8	9	7
1	2	3	4	5	6	7	8	9
3	1	2	6	4	5	9	7	8
8	9	7	2	3	1	5	6	4
7	8	9	1	2	3	4	5	6
9	7	8	3	1	2	6	4	5

1	3	2	7	9	8	4	6	5
3	2	1	9	8	7	6	5	4
2	1	3	8	7	9	5	4	6
7	9	8	4	6	5	1	3	2
9	8	7	6	5	4	3	2	1
8	7	9	5	4	6	2	1	3
4	6	5	1	3	2	7	9	8
6	5	4	3	2	1	9	8	7
5	4	6	2	1	3	8	7	9

- Corresponding rows are palindromes.
- Therefore, one square is a mirror-image of the other

Kim&Prasanna's doubly self-orthogonal Latin squares

- 병열 접속 네트워크 스위치 설계



Kichul Kim and Viktor K. Prasanna,
"**Latin Squares** for **Parallel Array Access**,"
IEEE Transactions and Parallel and Distributed Systems,
vol. 4, Issue 4, pp. 361-370, April 1993.

- **perfect** = both **Sudoku** and **double-diagonal**
- **self-orthogonal** = orthogonal to transpose
- **doubly self-orthogonal** = orthogonal to transpose and also to anti-transpose

Theorem

For all $\boldsymbol{n} = \boldsymbol{2^k m^2}, \boldsymbol{k \geq 2}$, $\boldsymbol{m}$ **odd**, there exists a **perfect** latin square of order n^2, which is also **doubly self-orthogonal**.

Choi's POLS of order 9 (1715, KOO-SOO-RYAK) for magic square

5	6	4	8	9	7	**2**	**3**	**1**
4	5	6	7	8	9	**1**	**2**	**3**
6	4	5	9	7	8	**3**	**1**	**2**
2	**3**	**1**	5	6	4	8	9	7
1	**2**	**3**	4	5	6	7	8	9
3	**1**	**2**	6	4	5	9	7	8
8	9	7	**2**	**3**	**1**	5	6	4
7	8	9	**1**	**2**	**3**	4	5	6
9	7	8	**3**	**1**	**2**	6	4	5

1	3	2	7	9	8	**4**	**6**	**5**
3	2	1	9	8	7	**6**	**5**	**4**
2	1	3	8	7	9	**5**	**4**	**6**
7	**9**	**8**	4	6	5	1	3	2
9	**8**	**7**	6	5	4	3	2	1
8	**7**	**9**	5	4	6	2	1	3
4	6	5	**1**	**3**	**2**	7	9	8
6	5	4	**3**	**2**	**1**	9	8	7
5	4	6	**2**	**1**	**3**	8	7	9

palindromic pair

- **NOT Sudoku**
- **singly diagonal (not doubly)**
- **Leads to a magic square by the canonical map**

Kim and Prasanna's POLS of order 9 (1993, IEEE Trans P.D.S) for parallel access

0	1	2	3	4	5	6	7	8
3	4	5	6	7	8	0	1	2
6	7	8	0	1	2	3	4	5
2	0	1	5	3	4	8	6	7
5	3	4	8	6	7	2	0	1
8	6	7	2	0	1	5	3	4
1	2	0	4	5	3	7	8	6
4	5	3	7	8	6	1	2	0
7	8	6	1	2	0	4	5	3

0	3	6	2	5	8	1	4	7
1	4	7	0	3	6	2	5	8
2	5	8	1	4	7	0	3	6
3	6	0	5	8	2	4	7	1
4	7	1	3	6	0	5	8	2
5	8	2	4	7	1	3	6	0
6	0	3	8	2	5	7	1	4
7	1	4	6	0	3	8	2	5
8	2	5	7	1	4	6	0	3

symmetric pair

perfect {
- **Sudoku**
  - *Any 3 x 3 main window is a permutation.*
- **doubly diagonal**

}
- **Leads to a magic square by the canonical map**

PROOF that

Choi's and K&P's are essentially the same

Start with this **(CHOI)**

5	6	4	8	9	7	2	3	1
4	5	6	7	8	9	1	2	3
6	4	5	9	7	8	3	1	2
2	3	1	5	6	4	8	9	7
1	2	3	4	5	6	7	8	9
3	1	2	6	4	5	9	7	8
8	9	7	2	3	1	5	6	4
7	8	9	1	2	3	4	5	6
9	7	8	3	1	2	6	4	5

4	5	3	7	8	6	1	2	0
3	4	5	6	7	8	0	1	2
5	3	4	8	6	7	2	0	1
1	2	0	4	5	3	7	8	6
0	1	2	3	4	5	6	7	8
2	0	1	5	3	4	8	6	7
7	8	6	1	2	0	4	5	3
6	7	8	0	1	2	3	4	5
8	6	7	2	0	1	5	3	4

Row permutation:

0	1	2	3	4	5	6	7	8
3	4	5	6	7	8	0	1	2
6	7	8	0	1	2	3	4	5
2	0	1	5	3	4	8	6	7
5	3	4	8	6	7	2	0	1
8	6	7	2	0	1	5	3	4
1	2	0	4	5	3	7	8	6
4	5	3	7	8	6	1	2	0
7	8	6	1	2	0	4	5	3

Symbol substitution:

1 → 0
2 → 1
3 → 2
4 → 3
5 → 4
6 → 5
7 → 6
8 → 7
9 → 8

End up with this **(K&P)**

Thus, they are **NOT essentially different**!!!

최석정의 9차 직교라틴방진의 local 구조

5	6	4	8	9	7	2	3	1
4	5	6	7	8	9	1	2	3
6	4	5	9	7	8	3	1	2
2	3	1	5	6	4	8	9	7
1	2	3	4	5	6	7	8	9
3	1	2	6	4	5	9	7	8
8	9	7	2	3	1	5	6	4
7	8	9	1	2	3	4	5	6
9	7	8	3	1	2	6	4	5

1,2,3	**A**
4,5,6	**B**
7,8,9	**C**

B	C	A
A	B	C
C	A	B

최석정의 9차 직교라틴방진의 local 구조

1	3	2	7	9	8	4	6	5
3	2	1	9	8	7	6	5	4
2	1	3	8	7	9	5	4	6
7	9	8	4	6	5	1	3	2
9	8	7	6	5	4	3	2	1
8	7	9	5	4	6	2	1	3
4	6	5	1	3	2	7	9	8
6	5	4	3	2	1	9	8	7
5	4	6	2	1	3	8	7	9

1,2,3	A
4,5,6	B
7,8,9	C

A	C	B
C	B	A
B	A	C

최석정의 9차 직교라틴방진의 local 구조

5_{1}	6_{3}	4_{2}	8_{7}	9_{9}	7_{8}	2_{4}	3_{6}	1_{5}
4_{3}	5_{2}	6_{1}	7_{9}	8_{8}	9_{7}	1_{6}	2_{5}	3_{4}
6_{2}	4_{1}	5_{3}	9_{8}	7_{7}	8_{9}	3_{5}	1_{4}	2_{6}
2_{7}	3_{9}	1_{8}	5_{4}	6_{6}	4_{5}	8_{1}	9_{3}	7_{2}
1_{9}	2_{8}	3_{7}	4_{6}	5_{5}	6_{4}	7_{3}	8_{2}	9_{1}
3_{8}	1_{7}	2_{9}	6_{5}	4_{4}	5_{6}	9_{2}	7_{1}	8_{3}
8_{4}	9_{6}	7_{5}	2_{1}	3_{3}	1_{2}	5_{7}	6_{9}	4_{8}
7_{6}	8_{5}	9_{4}	1_{3}	2_{2}	3_{1}	4_{9}	5_{8}	6_{7}
9_{5}	7_{4}	8_{6}	3_{2}	1_{1}	2_{3}	6_{8}	4_{7}	5_{9}

B_{A}	C_{C}	A_{B}
A_{C}	B_{B}	C_{A}
C_{B}	A_{A}	B_{C}

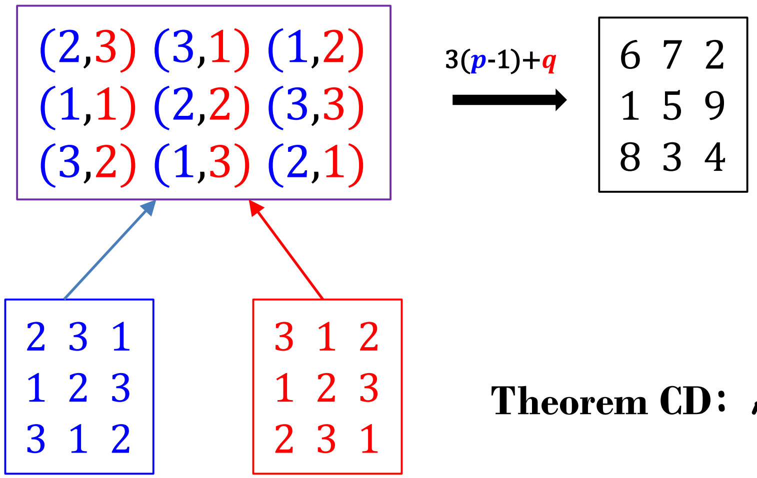
3차 직교라틴방진

1,2,3	A
4,5,6	B
7,8,9	C

B	C	A
A	B	C
C	A	B

A	*C*	*B*
C	*B*	*A*
B	*A*	*C*

(3) 임의의 POLS로부터 만들어지는 준마방진은 적당한 행/열 교환으로 마방진이 될 수 있는가



Theorem CD: Any **POLS** constructs a **semi-magic square** by the canonical map $\boldsymbol{n(p-1)+q}$.

Proof: For any row or column,

row-sum $= \sum_p n(p-1) + \sum_q q =$ column-sum

$= n\frac{n(n-1)}{2} + \frac{n(n+1)}{2} = \frac{n(n^2+1)}{2}$ **=** magic constant

Question: What about two diagonal sums?
Will it work for any pair of orthogonal Latin squares of order n ?

Sufficient condition for a magic square:

either

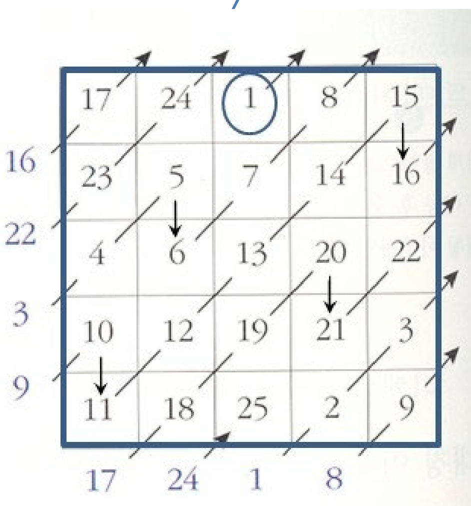
(Suff 1 all) both latin squares are **double-diagonal**

or

(Suff 2 odd) n=odd and **one diagonal is constant** (n+1)/2 and **the other diagonal is a permutation**, for both squares

They are NOT necessary!

$MS(5) = 5(p-1)+q$



4	5	1	2	3
5	1	2	3	4
1	2	3	4	5
2	3	4	5	1
3	4	5	1	2

2	4	1	3	5
3	5	2	4	1
4	1	3	5	2
5	2	4	1	3
1	3	5	2	4

Counter-example that
the sufficient condition in **Theorem CD** is **not necessary**.

Only one square has **one constant diagonal** of value (n+1)/2
And **all the other three diagonals are permutations** of 1,2,…,n

Sufficient conditions that the canonical map constructs a magic square

(Suff 2 odd)

3				1
	3		2	
		3		
	4		3	
5				3

and

2				3
	5		3	
		3		
	3		1	
3				4

or

(Suff 1 all)

1				4
	2		1	
		3		
	5		4	
2				5

and

1				2
	2		5	
		3		
	1		4	
4				5

(Suff 2 odd)

3	5	2	4	1
1	3	5	2	4
4	1	3	5	2
2	4	1	3	5
5	2	4	1	3

and

2	1	5	4	3
1	5	4	3	2
5	4	3	2	1
4	3	2	1	5
3	2	1	5	4

(Suff 1 all)

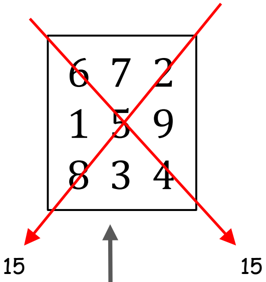
1	3	5	2	4
5	2	4	1	3
4	1	3	5	2
3	5	2	4	1
2	4	1	3	5

and

1	5	4	3	2
3	2	1	5	4
5	4	3	2	1
2	1	5	4	3
4	3	2	1	5

Theorem [CD, 3.108]: Two orthogonal double-diagonal latin squares of order n exists **if and only if** $n \neq 2, 3,$ or 6.

First Example



magic !

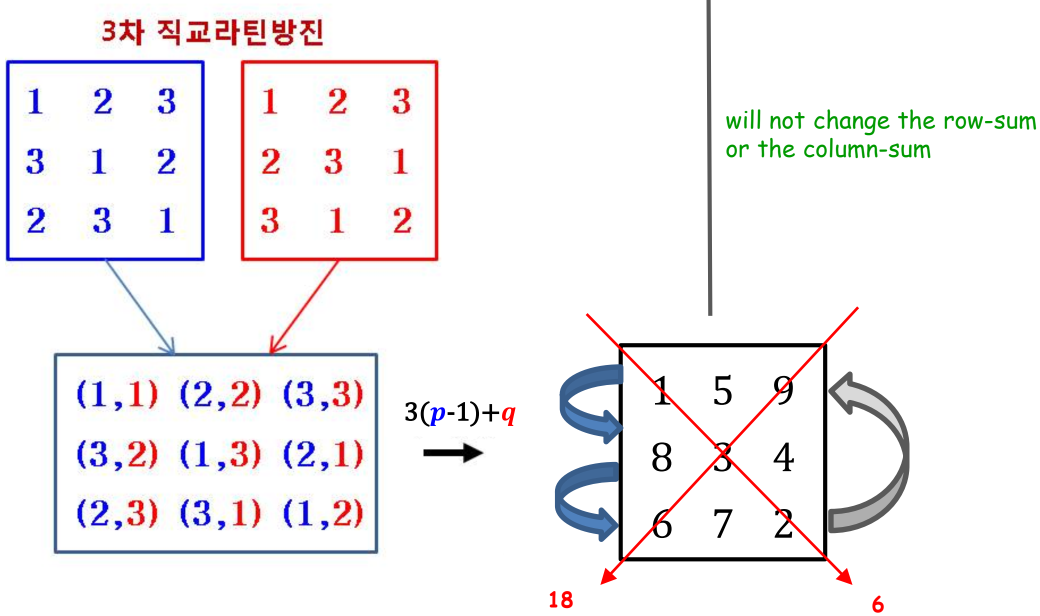
The semi-magic of order 3 from **any** 3x3 POLS by the canonical map has this property?

I guess so, simply because 3 is too small.

(Conjecture)
Any semi-magic square of order 3 from a 3x3 POLS by the canonical map can be transformed into a magic square by some combination of row/column permutations

move the bottom row to the top

Observe:



Consider another example of POLS of order 4.

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

1	2	3	4
3	4	1	2
4	3	2	1
2	1	4	3

1 1	2 2	3 3	4 4
2 3	1 4	4 1	3 2
3 4	4 3	1 2	2 1
4 2	3 1	2 4	1 3

1	**6**	**11**	**16**
7	**4**	**13**	**10**
12	**15**	**2**	**5**
14	**9**	**8**	**3**

column-sum = row-sum = **34** = magic constant

1diagonal-sum = 1+4+2+3 = **10**

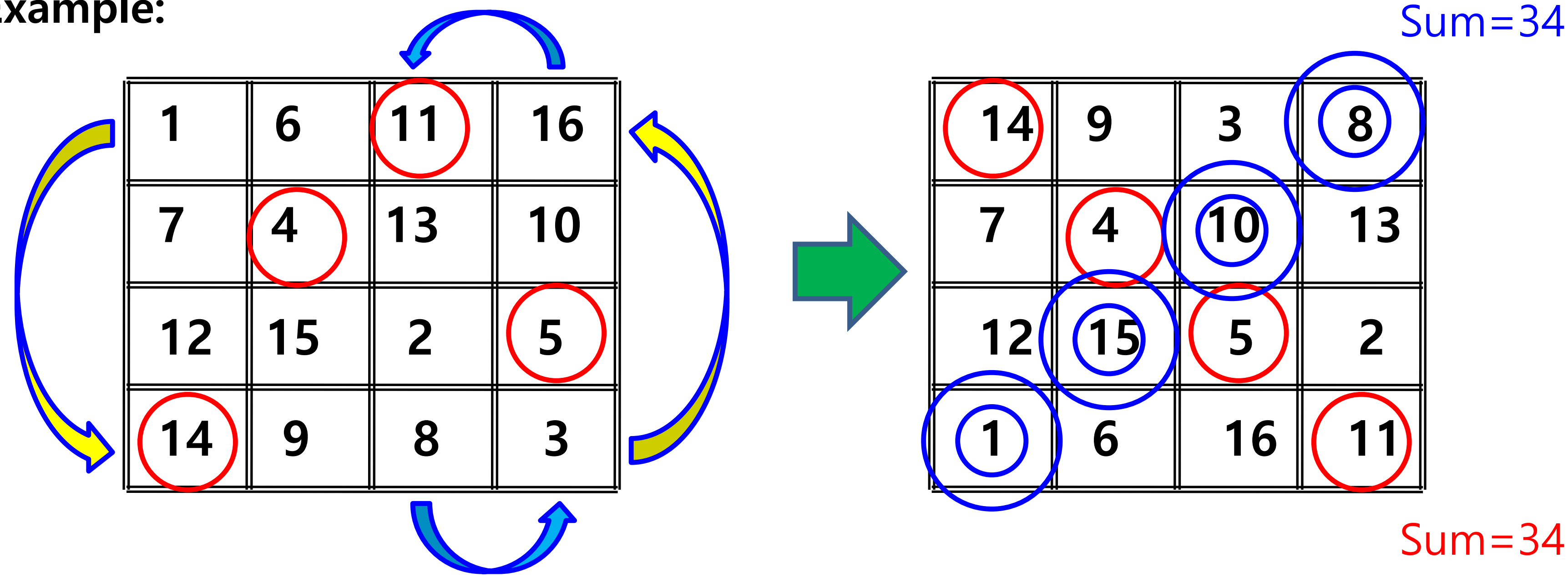
2diagonal-sum = 13+14+15+16 = **58**

It is only a <u>semi</u>-magic square!!!

Some observations:

Any **row permutation** or **column permutation** will not change the row-sum or the column-sum of a semi-magic square.

Example:



Main Question Q1 (in terms of semi-magic)

Is it true that the semi-magic square (from a POLS by the canonical map) can be transformed into a magic square by some combination of row/column permutations?

- We need both diagonal sums to be the magic constant.

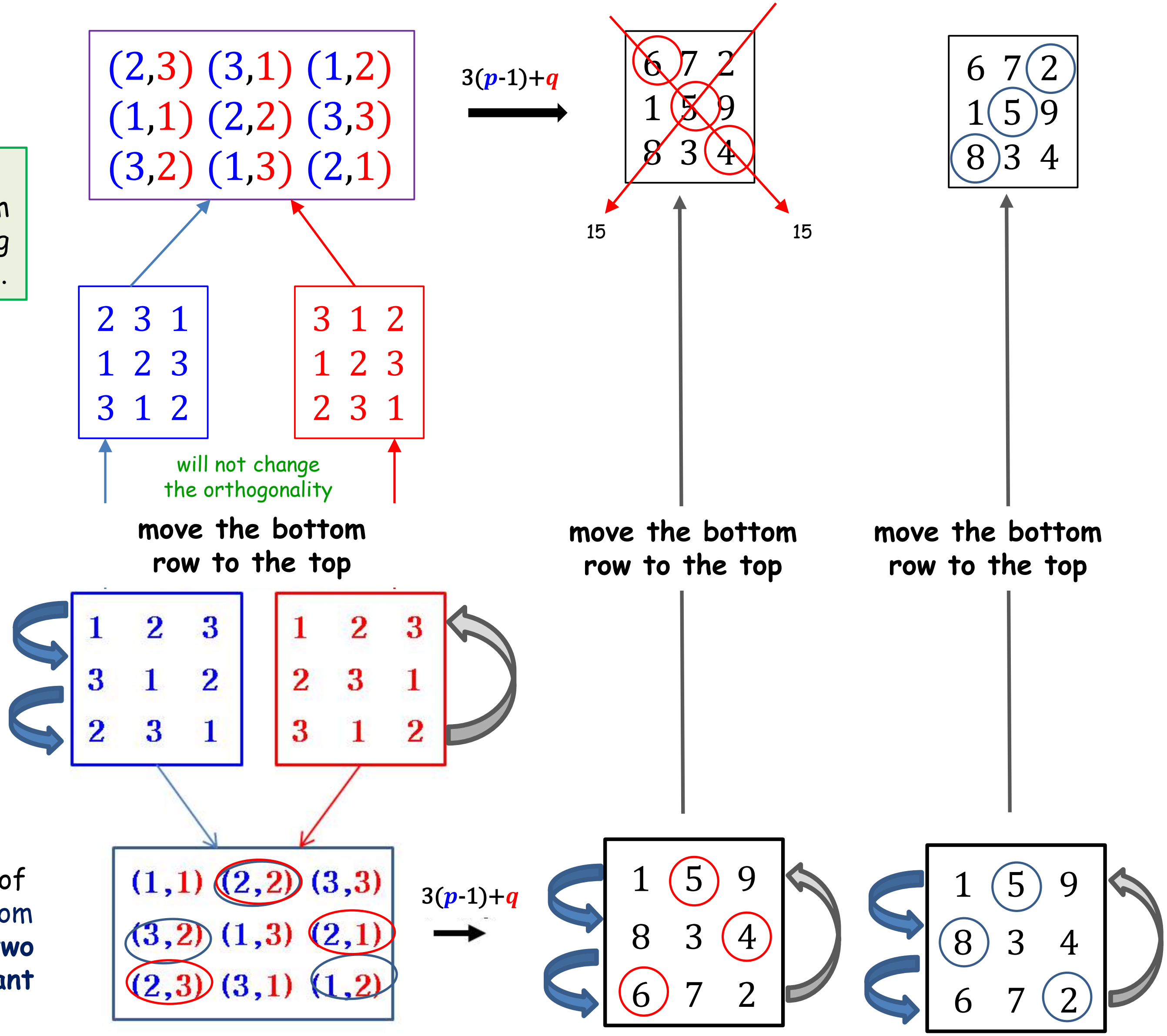
- 그런데 이런 문제를 풀기 위해서 어떻게 접근해야할까…
- Combinatorial mathematics 문제들은 항상 이렇게 시작됩니다…

- canonical map으로 만들어진 semi-magic square 직전의 상황으로 돌아가서 살펴볼까

one step backward

The situation in the POLS before the canonical construction

A **transversal** in a latin square is a set of n positions, one from each row and column, containing each of n symbols exactly once.



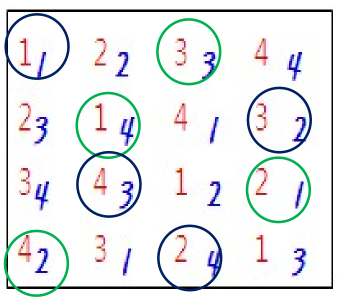
symmetric pattern of two **transversals** from both latin squares, **two of which are constant** (n+1)/2 = 2

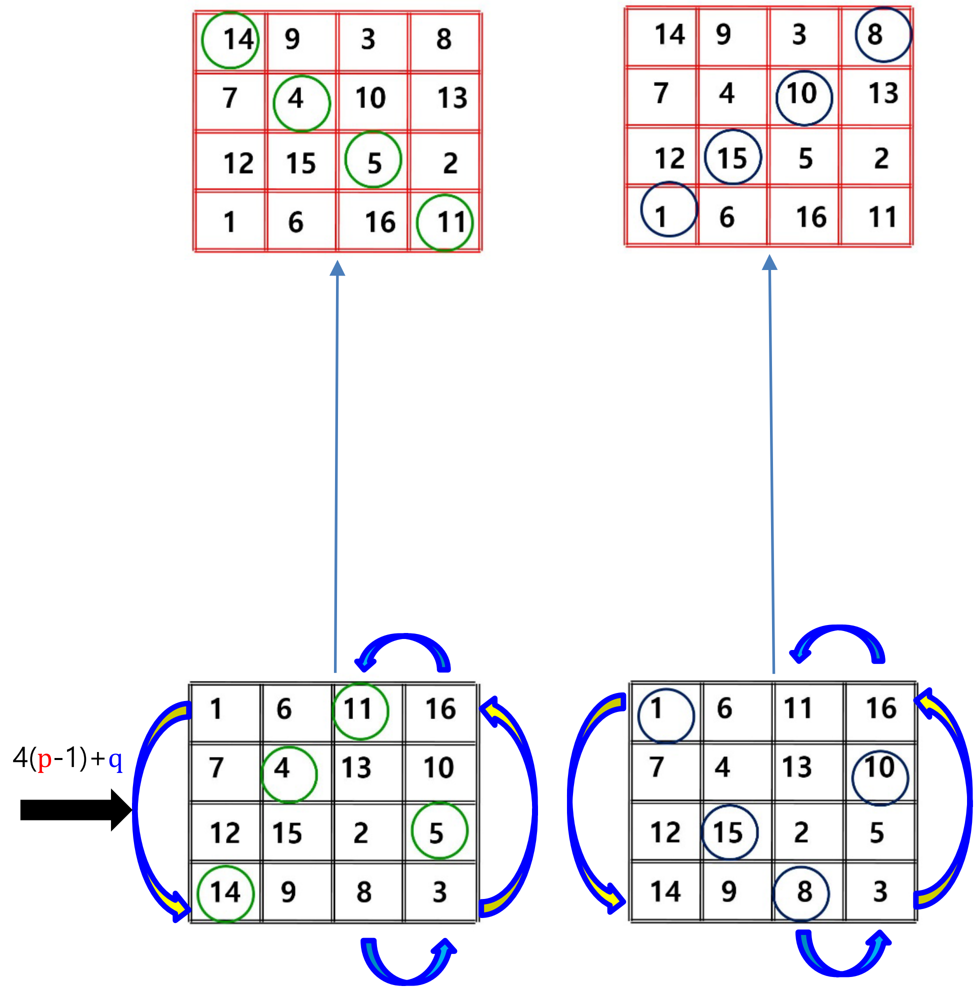
...(Suff 2 odd)...

one step backward

The situation in the POLS before the canonical construction

A **transversal** in a latin square is a set of n positions, one from each row and column, containing each of n symbols exactly once.





- the **same set of positions** of **RED** and **BLUE** latin squares, respectively.
- **symmetric** patterns of two **transversals** from both latin squares

...(Suff 1 all)...

Main Question (Q1) Again (in terms of POLS):

Is it true that, for **any** POLS,

(1) when n=even or odd, they have a **symmetric** pattern of **two transversals** in **both** squares at the **same** set of n positions; and/or

(2) when n=odd, **two** of **such four transversal positions** contain **a constant**?

Seems to be still helpless… 여전히 실마리 찾기가 어려움…
How to approach these problems?

What if we consider **only one** diagonal sum? (instead of both diagonal sums)

Another [weaker] Question (Q2)

What if we care **only** of **one** diagonal sum?

We have to consider the existence of only one transversal of each square at the same n positions.

The question on POLS becomes the following:

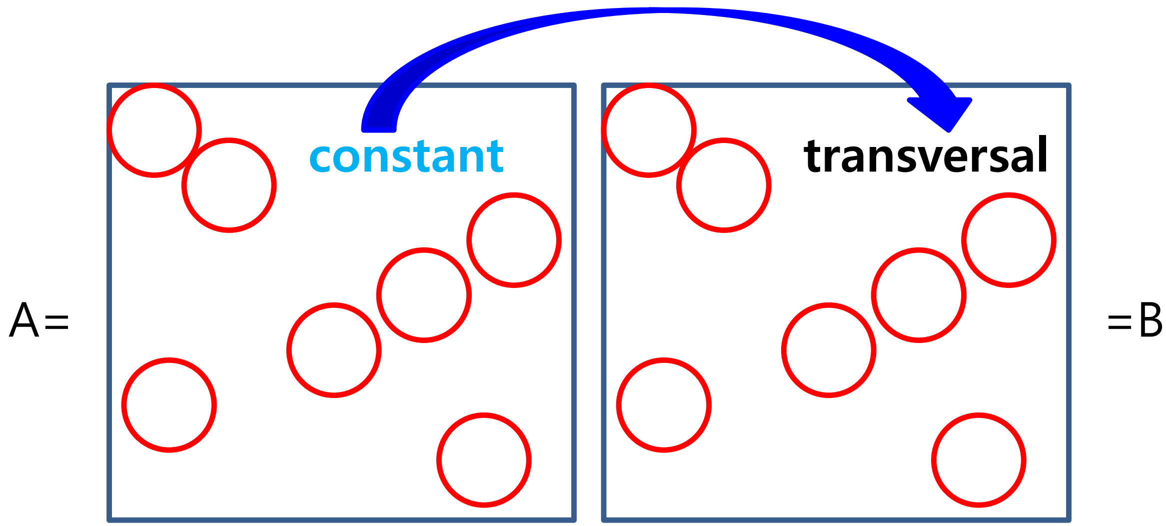
Is it true that, for **any** POLS,

(1) when **n=even or odd**, they have **a transversal** in **both** squares at the **same** set of **n positions**; and/or

(2) when **n=odd**, **a set of n positions** containing a constant in one square **is a transversal** in the other square?

Observation (for the second part of Q2):

Given **ANY** POLS, A and B, of order n **(odd)** and any n cells of A containing **a constant**, the n positions in B corresponding to these n cells in A must be **a transversal** in B.

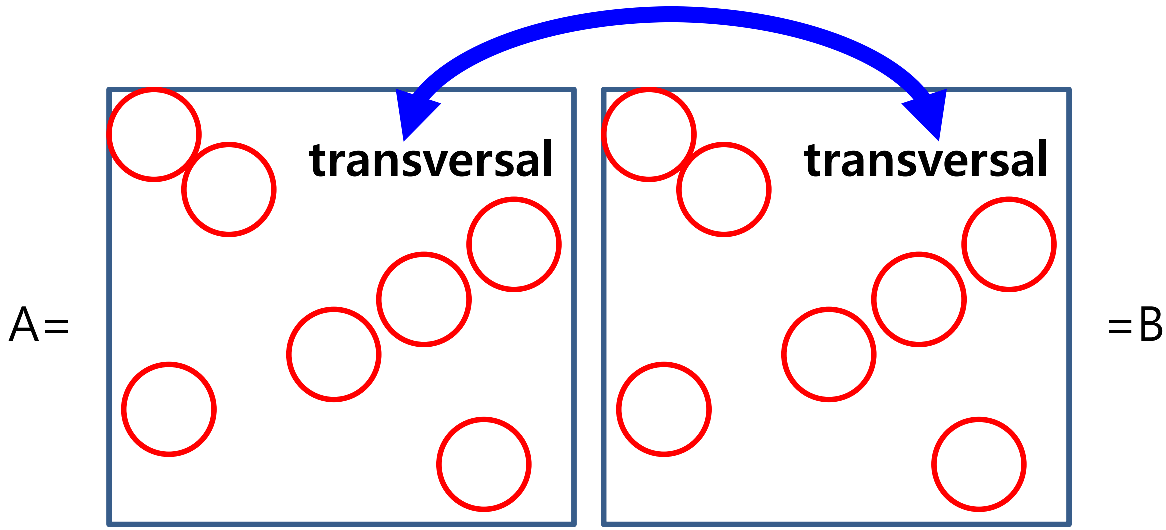


Remark: From the above observation, **when n=odd**, any given POLS can be transformed by a combination of row/column permutations into a form that leads to a semi-magic square in which **at least one of two diagonal sums is the magic constant**.

-- Proof that (2) of Q2 is TRUE.

CONJECTURE 1 for (1) of Q2 :

Any POLS A and B of order n have transversals at the same n positions



Conjecture:

any POLS can be transformed by a combination of row/column permutations into a form that leads to a semi-magic square in which at least **one of two diagonal sums is the magic constant**.

CONJECTURE 1 for (1) of Q2 :

POLS A and B of order n have transversals at the same n positions

Turned out to be **false** for odd order by the following:

No transversals in the **same** set of 9 positions in both squares.

Choi's POLS of order 9=odd

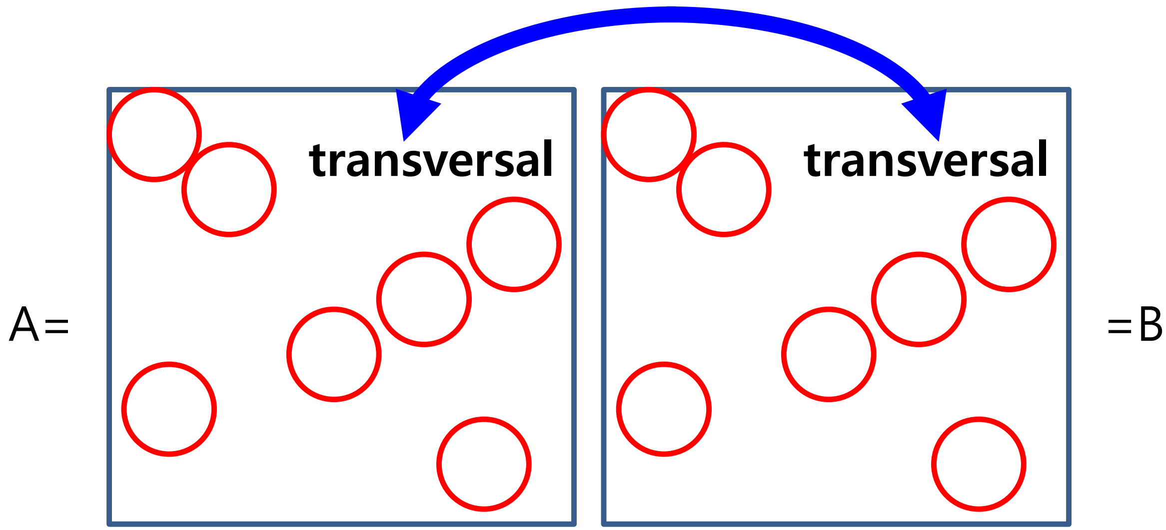
(1715, KOO-SOO-RYAK)

5	6	4	8	9	7	2	3	1
4	5	6	7	8	9	1	2	3
6	4	5	9	7	8	3	1	2
2	3	1	5	6	4	8	9	7
1	2	3	4	5	6	7	8	9
3	1	2	6	4	5	9	7	8
8	9	7	2	3	1	5	6	4
7	8	9	1	2	3	4	5	6
9	7	8	3	1	2	6	4	5

1	3	2	7	9	8	4	6	5
3	2	1	9	8	7	6	5	4
2	1	3	8	7	9	5	4	6
7	9	8	4	6	5	1	3	2
9	8	7	6	5	4	3	2	1
8	7	9	5	4	6	2	1	3
4	6	5	1	3	2	7	9	8
6	5	4	3	2	1	9	8	7
5	4	6	2	1	3	8	7	9

CONJECTURE 2 -- (1) of Q2 for **n even:**

Any POLS of an **even** order has **a transversal** in both squares at the **same** set of n positions.



If this is ture, then, **when n=even**, any given POLS can be transformed by a combination of row/column permutations into a form that leads to a semi-magic square in which **at least one of two diagonal sums is the magic constant**.

Modified Question: What is the condition on POLS (of even order n) so that both have a transversal at the same set of n positions?

Easy and straightforward (even or odd) sufficient condition :

whenever the **POLS** have a **THIRD** orthogonal **mate**.

The n positions in third orthogonal mate with a constant become a transversal in both of the original POLS.

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

1	2	3	4
3	4	1	2
4	3	2	1
2	1	4	3

1	2	3	4
4	3	2	1
2	1	4	3
3	4	1	2

So, if the conjecture is not true, then a counterexample could be a POLS that do not have any third orthogonal mate. [further necessary condition]

First Try: exhaustive search for the order 4.

RESULT: Every pair is a member of an orthogonal **triplet**. – **no counterexample**

For order 6, there does not exist a PAIR !
For order 8, there are too many for an exhaustive search.
For order 10 ??

Conjecture 2 turned out to be **FALSE** by the counterexample of order 10 below.

For order 10, we checked with a POLS by **Parker (1959)** and found that

- they do not possess a third orthogonal mate **and further**
- no such a pair of transversals exists. [may need a double check!!!]

$$
\begin{pmatrix}
1 & 7 & 6 & 5 & 0 & 9 & 8 & 2 & 3 & 4 \\
8 & 2 & 1 & 7 & 6 & 0 & 9 & 3 & 4 & 5 \\
9 & 8 & 3 & 2 & 1 & 7 & 0 & 4 & 5 & 6 \\
0 & 9 & 8 & 4 & 3 & 2 & 1 & 5 & 6 & 7 \\
2 & 0 & 9 & 8 & 5 & 4 & 3 & 6 & 7 & 1 \\
4 & 3 & 0 & 9 & 8 & 6 & 5 & 7 & 1 & 2 \\
6 & 5 & 4 & 0 & 9 & 8 & 7 & 1 & 2 & 3 \\
3 & 4 & 5 & 6 & 7 & 1 & 2 & 8 & 0 & 9 \\
5 & 6 & 7 & 1 & 2 & 3 & 4 & 0 & 9 & 8 \\
7 & 1 & 2 & 3 & 4 & 5 & 6 & 9 & 8 & 0
\end{pmatrix}
\quad
\begin{pmatrix}
1 & 8 & 9 & 0 & 2 & 4 & 6 & 3 & 5 & 7 \\
7 & 2 & 8 & 9 & 0 & 3 & 5 & 4 & 6 & 1 \\
6 & 1 & 3 & 8 & 9 & 0 & 4 & 5 & 7 & 2 \\
5 & 7 & 2 & 4 & 8 & 9 & 0 & 6 & 1 & 3 \\
0 & 6 & 1 & 3 & 5 & 8 & 9 & 7 & 2 & 4 \\
9 & 0 & 7 & 2 & 4 & 6 & 8 & 1 & 3 & 5 \\
8 & 9 & 0 & 1 & 3 & 5 & 7 & 2 & 4 & 6 \\
2 & 3 & 4 & 5 & 6 & 7 & 1 & 8 & 9 & 0 \\
3 & 4 & 5 & 6 & 7 & 1 & 2 & 0 & 8 & 9 \\
4 & 5 & 6 & 7 & 1 & 2 & 3 & 9 & 0 & 8
\end{pmatrix}
$$

Conclusion on Questions (Q2 and Q1)

Is it true that, for **any** POLS,

(1) when **n=even or odd**, they have **a transversal** in **both** squares at the **same** set of n positions; and/or

- **False in general by counter-example** for both even (10) and odd (9)
- Maybe true(?) for some infinitely many values of n greater than some number, say 10
- **Therefore, (1) of Q1 becomes false.**

- **One sufficient condition** is when the POLS is a member of **an orthogonal triplet**.

(2) when **n=odd**, **a set of n positions** containing a constant in one square **is a transversal** in the other square?

- **(2) of Q2 is True in general.**
- **(2) of Q1** <for both diagonal sums> **is still open.**

Summary

1. **Choi Seok-Jeong in Korea has constructed a pair of orthogonal latin squares (POLS) of order 9 in 1715 or earlier, which is <u>at least 61 years earlier</u> than that of Euler's.**
   - This was appeared in "Handbook of Combinatorial Designs" 2nd ed., edited by C. Colbourn and J. Dinitz, 2007, published by Chapman Hall & CRC.
   - This POLS is **so special** that they construct a magic square of order 9.
   - This POLS is **essentially the same** as those by Kim and Prasanna in 1993 (also by myself).

2. **Question:** Every POLS can make a semi-magic square whose one diagonal sum is the magic constant by a combination of row/column permutations and the canonical map:

   (Q2.2) True for an **odd** order by **(Suff 2 odd) condition** in **Theorem CD**.

   (Q2.1) For **even/odd** orders, not every POLS has the same set of transversal positions

   - Counterexamples at order 9(최석정) and order 10(Parker)
   - Cannot make such a semi-magic **thru (Suff 1 all) condition** in **Theorem CD.**
   - **modified conjecture: (Q2.1) is true for all n > some n'**

3. **A sufficient condition** for **a** POLS of order n to construct a magic square with a combination of row/column permutations and the canonical map:

   Whenever the POLS has a third orthogonal mate.

   - **It is still open** whether this is also necessary.

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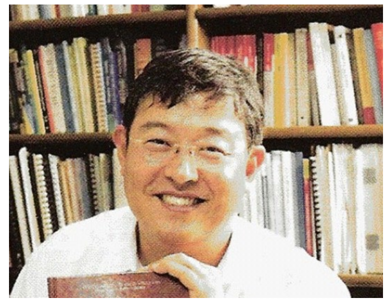
End

any question?



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Research Interest

Communication engineering:

Communication & Coding Theory,
GPS, Mobile Communications

Applied mathematics:

Pseudo-random Sequences, Cyclic difference sets,
Latin/Tuscan/Florentin squares