A construction of 2-sequential-recovery locally repairable codes

ICTC2021

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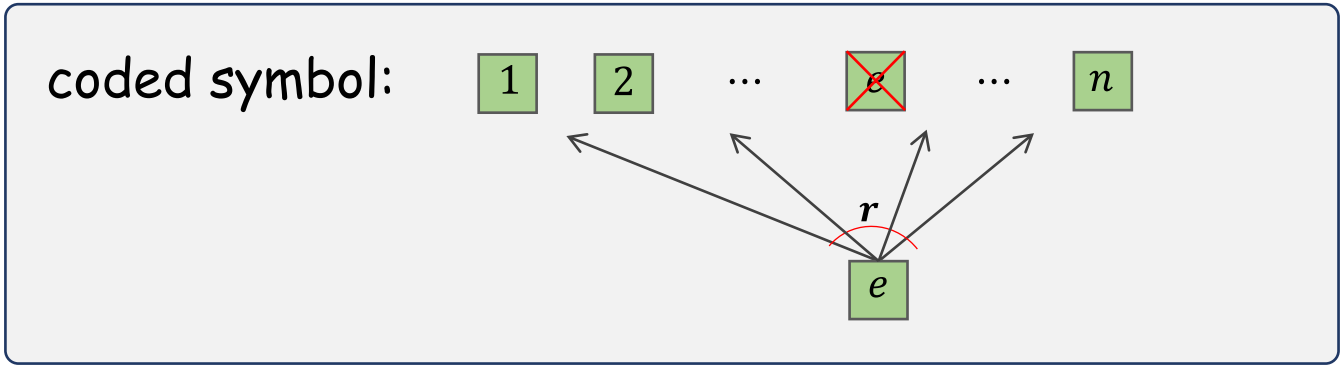
Locality

- For an $[n,k,d]$ linear code C with the parity check matrix H:



  - <u>Locality of symbol:</u>
    the (minimum) number of symbols needed to repair the erased symbol.
  - <u>Code C has locality r:</u>
    All coded symbols have the locality at most r.

C is denoted as $[n,k,d,r]$ LRC

Locality

- For an $[n, k, d]$ linear code C with the parity check matrix H:

  ✓ $c = (c_1, c_2, \dots, c_n)$ be a nonzero codeword
  ✓ $h_i = (h_{i,1}, h_{i,2}, \dots \dots, h_{i,n})$ be the i^{th} row of H.

$$\boldsymbol{c_1 h_{i,1} + c_2 h_{i,2} + c_3 h_{i,3} + \cdots c_n h_{i,n} = 0}$$

Locality

- For an $[n, k, d]$ linear code C with the parity check matrix H:

  ✓ $c = (c_1, c_2, \ldots, c_n)$ be a nonzero codeword
  ✓ $h_i = (\underbrace{h_{i,1}, h_{i,2},}_{\neq 0} \underbrace{\ldots \ldots, h_{i,n}}_{= 0})$ be the i^{th} row of H.

$$\cancel{\boldsymbol{c_1 h_{i,1} + c_2 h_{i,2} + c_3 h_{i,3} + \cdots c_n h_{i,n} = 0}}$$

$$\boldsymbol{c_1 h_{i,1} + c_2 h_{i,2} + c_3 h_{i,3} + \cdots c_{r+1} h_{i,r+1} = 0}$$

Locality

- For an $[n, k, d]$ linear code C with the parity check matrix H:

  ✓ $c = (c_1, c_2, \ldots, c_n)$ be a nonzero codeword

  ✓ $h_i = (\underbrace{h_{i,1}, h_{i,2}, \ldots}_{\neq 0}\underbrace{\ldots, h_{i,n}}_{= 0})$ be the i^{th} row of H.

$$\cancel{\boldsymbol{c_1 h_{i,1} + c_2 h_{i,2} + c_3 h_{i,3} + \cdots c_n h_{i,n} = 0}}$$

$$\boldsymbol{c_1 h_{i,1} + c_2 h_{i,2} + c_3 h_{i,3} + \cdots \cancel{c_{r+1}} h_{i,r+1} = 0}$$

the repair set of c_{r+1}: $\boldsymbol{R(r+1) = \{1, 2, \ldots, r\}}$

Locality

- For an $[n, k, d]$ linear code C with the parity check matrix H satisfying:
  - Each row of H has at most $r + 1$ nonzero elements
  - No all-zero column in H

$$\Longrightarrow \text{An } [n, k, d, r] \text{ LRC } C$$

Example:

$$
\begin{array}{c c}
 & \begin{array}{ccccccccccc} \overbrace{}^{r+1} & \overbrace{}^{r+1} & \cdots & \overbrace{}^{r+1} \end{array} \\
H = & \begin{bmatrix} \neq 0 \quad \dots \quad \neq 0 & & & \\ & \neq 0 \quad \dots \quad \neq 0 & & \\ & & \ddots & \\ & & & \neq 0 \quad \dots \quad \neq 0 \end{bmatrix}
\end{array}
$$

Example of sequential-recovery

- For an $[n, k, d, r]$ LRC C with the parity check matrix H as follows:

$$H = \begin{bmatrix} \overbrace{\neq 0 \quad \cdots \quad \boxed{\neq 0}}^{r+1} & & & \\ & \overbrace{\neq 0 \quad \cdots \quad \neq 0}^{r+1} & \overset{\cdots}{} & \\ & & \ddots & \\ & & & \overbrace{\neq 0 \quad \cdots \quad \neq 0}^{r+1} \end{bmatrix}$$

c_{r+1}

  - $\boldsymbol{R(r+1) = \{1, 2, \ldots, r\}}$

Example of sequential-recovery

- For an $[n, k, d, r]$ LRC C with the parity check matrix H as follows:

$$H = \begin{array}{c} \begin{array}{ccccccccc} \multicolumn{3}{c}{\overbrace{\hphantom{\neq 0 \quad \cdots \quad \neq 0}}^{r+1}} & \multicolumn{3}{c}{\overbrace{\hphantom{\neq 0 \quad \cdots \quad \neq 0}}^{r+1}} & \cdots & \multicolumn{2}{c}{\overbrace{\hphantom{\neq 0 \quad \cdots \quad \neq 0}}^{r+1}} \end{array} \\ \begin{bmatrix} \boxed{\neq 0} & \cdots & \boxed{\neq 0} & & & & & & & \\ & & & \neq 0 & \cdots & \neq 0 & & & & \\ & & & & & & \ddots & & & \\ & & & & & & & \neq 0 & \cdots & \neq 0 \end{bmatrix} \\ \begin{array}{ccc} c_1 & \qquad & c_{r+1} \end{array} \end{array}$$

  - $\boldsymbol{R(r+1) = \{\cancel{1}, 2, \ldots, r\}}$

Example of sequential-recovery

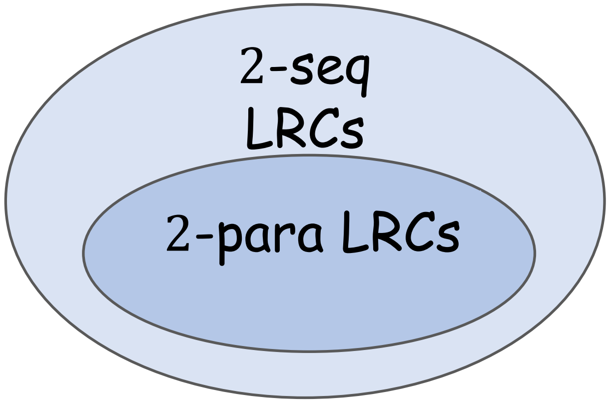
- For an $[n, k, d, r]$ LRC C with the parity check matrix H as follows:

$$H = \begin{array}{c} \begin{bmatrix} \neq 0 & \cdots & \neq 0 & & & \\ & & \neq 0 & \cdots & \neq 0 & \\ & & & & & \ddots \end{bmatrix} \\ \begin{array}{ccc} c_1 & \quad\quad & c_{r+1} \end{array} \end{array}$$

$$\overbrace{\neq 0 \ \cdots \ \neq 0}^{r+1} \qquad \underbrace{\neq 0 \ \cdots \ \neq 0}_{r+1}$$

  - First, repair c_{r+1};
  - Then, repair c_1 using c_{r+1}.

2-sequential-recovery (2-seq) LRCs [5]



Let C be an $[n, k, d, r]$ LRC.

For any two erasures e_x and e_y,

if e_x and e_y satisfy any of the condition,

Condition	Recovery order	
	1^{st}	2^{nd}
$\boldsymbol{y \notin R(x)}$	e_x	e_y
$\boldsymbol{x \notin R(y)}$	e_y	e_x

then, C is a 2-seq $[n, k, d, r]$ LRCs.

[5] N. Prakash, V. Lalitha, S. B. Balaji, P. V. Kumar, "Codes with locality for two erasures," IEEE Trans. Inf. Theory, vol. 65, no. 12, pp. 7771-7789, Dec. 2019.

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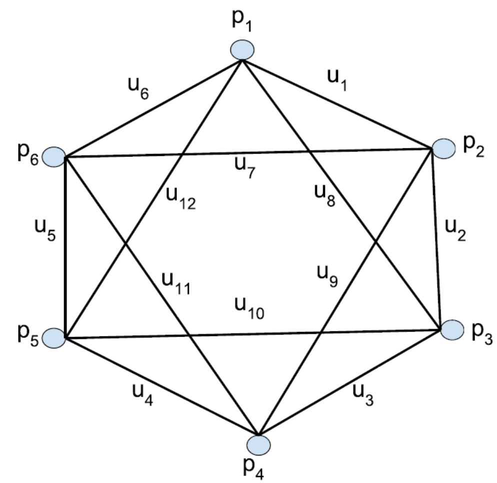
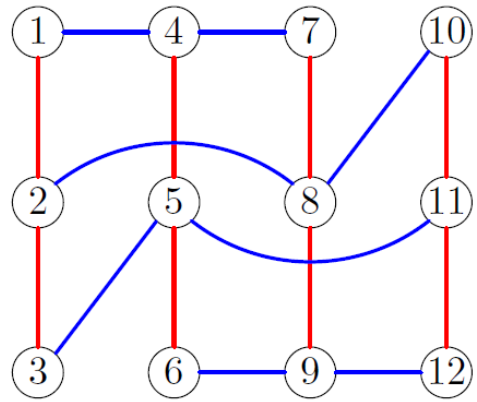
Existed 2-seq LRCs



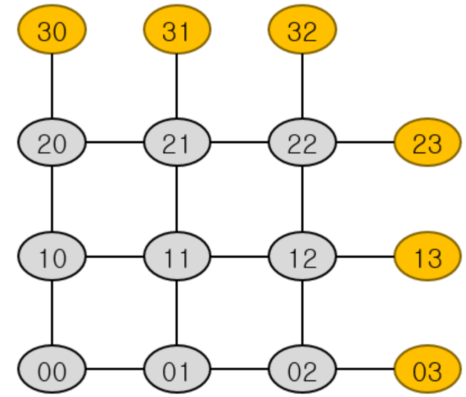
LRC $\mathcal{C}$ [5]:

$$\left[\frac{(r+2)k}{r}, \geq \frac{r(r+1)}{2}, r\right]$$



LRC $\mathcal{C}$ [6]:

$$\left[\frac{(r+2)k}{r}, k, r\right]$$



LRC $\mathcal{C}$ [7]:

$$\left[\left(1+\frac{2}{r}\right)r^m, r^m, r\right]$$

[5] N. Prakash, V. Lalitha, S. B. Balaji, P. V. Kumar, "Codes with locality for two erasures," IEEE Trans. Inf. Theory, vol. 65, no. 12, pp. 7771-7789, Dec. 2019.

[6] W. Song and C. Yuen, "Locally repairable codes with functional repair and multiple erasure tolerance," Jul. 2015, arXiv:1507.02796. [Online]. Available: https://arxiv.org/abs/1507.02796

[7] W. Song, K. Cai, C. Yuen, K. Cai, and G. Han, "On sequential locally repairable codes," IEEE Trans. Inf. Theory, vol. 64, no. 5, pp. 3513–3527, May 2018.

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New Construction of 2-seq LRCs

Construction 1:

Let $r > 1$ be any integer. Let q be a prime power such that $q \geq r+1$, and α be the primitive element of $\mathbb{F}_q$. Let m be any positive integer. The code C has the parity check matrix as follows

$$H = \overbrace{\begin{bmatrix} H_r & 0 & \cdots & 0 \\ 0 & H_r & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & H_r \end{bmatrix}}^{m},$$

where

$$H_r = \begin{bmatrix} 1 & 1 & \dots & 1 & 1 & 0 \\ \alpha & \alpha^2 & \dots & \alpha^r & 0 & 1 \end{bmatrix}.$$

New Construction

Theorem 1:
The code C from Construction 1 is a 2-seq $[(r+2)m, rm, 3, r]$ LRC over $\mathbb{F}_q$.

The proof is omitted.

Corollary 1:
When $m = 1$, the 2-seq $[r+2, r, 3, r]$ LRC is the maximum distance separable (MDS) code.

$$d \leq n - k + 1 = 3.$$

Example

A 2-seq $[10,6,3,3]$ LRC over $\mathbb{F}_{2^2}$ has the parity check matrix:

$$\begin{array}{c} \overbrace{\hspace{12em}}^{r = 3 + 2} \hspace{14em} \end{array}$$

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha & \alpha^2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha & \alpha^2 & 1 & 0 & 1 \end{pmatrix} \qquad m = 2$$

$$c = \begin{pmatrix} \cancel{1} & \alpha & \alpha^2 & \cancel{0} & 0 & 1 & \alpha & \alpha^2 & 0 & 0 \end{pmatrix}$$

Example

A 2-seq $[10,6,3,3]$ LRC over $\mathbb{F}_{2^2}$ has the parity check matrix:

$$r = 3 + 2$$

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha & \alpha^2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha & \alpha^2 & 1 & 0 & 1 \end{pmatrix} \quad m = 2$$

$$c = (\cancel{1} \;\; \alpha \;\; \alpha^2 \;\; \cancel{0} \;\; 0 \;\; 1 \;\; \alpha \;\; \alpha^2 \;\; 0 \;\; 0)$$

Example

A 2-seq $[10,6,3,3]$ LRC over $\mathbb{F}_{2^2}$ has the parity check matrix:

$$
\begin{array}{cc}
 & \begin{array}{cccccccccc} \multicolumn{5}{c}{\overbrace{\hphantom{1 \quad 1 \quad 1 \quad 1 \quad 0}}^{r = 3 + 2}} & & & & & \end{array} \\
H = & \left(\begin{array}{ccccc|ccccc}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\alpha & \alpha^2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & \alpha & \alpha^2 & 1 & 0 & 1
\end{array}\right)
\end{array}
\qquad m = 2
$$

$$c = (\cancel{1} \quad \alpha \quad \alpha^2 \quad \cancel{0} \quad 0 \quad 1 \quad \alpha \quad \alpha^2 \quad 0 \quad 0)$$

$$c = (1 \quad \alpha \quad \alpha^2 \quad \cancel{0} \quad 0 \quad 1 \quad \alpha \quad \alpha^2 \quad 0 \quad 0)$$

Example

A 2-seq $[10,6,3,3]$ LRC over $\mathbb{F}_{2^2}$ has the parity check matrix:

$$\begin{array}{c} \overbrace{\hphantom{1\quad 1\quad 1\quad 1\quad 0}}^{r = 3 + 2} \end{array}$$

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha & \alpha^2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha & \alpha^2 & 1 & 0 & 1 \end{pmatrix} \qquad m = 2$$

$$c = (\cancel{1} \quad \alpha \quad \cancel{\alpha^2} \quad 0 \quad 0 \quad 1 \quad \alpha \quad \alpha^2 \quad 0 \quad 0)$$

Example

A 2-seq $[10,6,3,3]$ LRC over $\mathbb{F}_{2^2}$ has the parity check matrix:

$$r = 3 + 2$$

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha & \alpha^2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha & \alpha^2 & 1 & 0 & 1 \end{pmatrix} \qquad m = 2$$

$$c = \begin{pmatrix} 1 & \alpha & \alpha^2 & 0 & 0 & 1 & \alpha & \alpha^2 & 0 & 0 \end{pmatrix}$$

$h_1 - h_2$:	$1 - \alpha$	$1 - \alpha^2$	0	1	-1	0	0	0	0	0
$h_2 - \alpha h_1$:	0	$\alpha^2 - \alpha$	$1 - \alpha$	$-\alpha$	1	0	0	0	0	0

Example

A 2-seq $[10,6,3,3]$ LRC over $\mathbb{F}_{2^2}$ has the parity check matrix:

$$H = \begin{pmatrix} \overbrace{1 & 1 & 1 & 1 & 0}^{r = 3 + 2} & 0 & 0 & 0 & 0 & 0 \\ \alpha & \alpha^2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha & \alpha^2 & 1 & 0 & 1 \end{pmatrix} \qquad m = 2$$

$$c = (\cancel{1} \;\; \alpha \;\; \cancel{\alpha^2} \;\; 0 \;\; 0 \;\; 1 \;\; \alpha \;\; \alpha^2 \;\; 0 \;\; 0)$$

$h_1 - h_2$:	$1-\alpha$	$1-\alpha^2$	0	1	-1	0	0	0	0	0
$h_2 - \alpha h_1$:	0	$\alpha^2-\alpha$	$1-\alpha$	$-\alpha$	1	0	0	0	0	0

Code rate and minimum distance of C

Theorem 2:
The 2-seq $[(r+2)m, rm, 3, r]$ LRCs from Construction 1 are rate-optimal and distance-optimal.

Rate bound:

$$\frac{k}{n} = \frac{r}{r+2};$$

Minimum distance:

$$d = 3.$$

[5] N. Prakash, V. Lalitha, S. B. Balaji, P. V. Kumar, "Codes with locality for two erasures," IEEE Trans. Inf. Theory, vol. 65, no. 12, pp. 7771-7789, Dec. 2019.

Comparison of the existed 2-seq LRCs

TABLE I
COMPARISON OF THE RATE-OPTIMAL 2-SEQ LRCs

Reference	Parameters			Field Size
	n	k	r	
This paper	$(r+2)m$	mr	any r	$\geq r+1$
[5]	$\frac{(r+2)k}{r}$	$\geq \frac{r(r+1)}{2}$	any r	2
[6]	$\frac{(r+2)k}{r}$	$\lceil \frac{k}{r} \rceil \geq r$	any r	2
[7]	$(1+\frac{2}{r})r^m$	r^m	any r	2

- The choice of k is more flexible;
- New perspective of construction of 2-seq LRCs.

[5] N. Prakash, V. Lalitha, S. B. Balaji, P. V. Kumar, "Codes with locality for two erasures," IEEE Trans. Inf. Theory, vol. 65, no. 12, pp. 7771-7789, Dec. 2019.

[6] W. Song and C. Yuen, "Locally repairable codes with functional repair and multiple erasure tolerance," Jul. 2015, arXiv:1507.02796. [Online]. Available: https://arxiv.org/abs/1507.02796

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Thank you !