Some Intersections of two Binary LRCs with Disjoint Repair Groups

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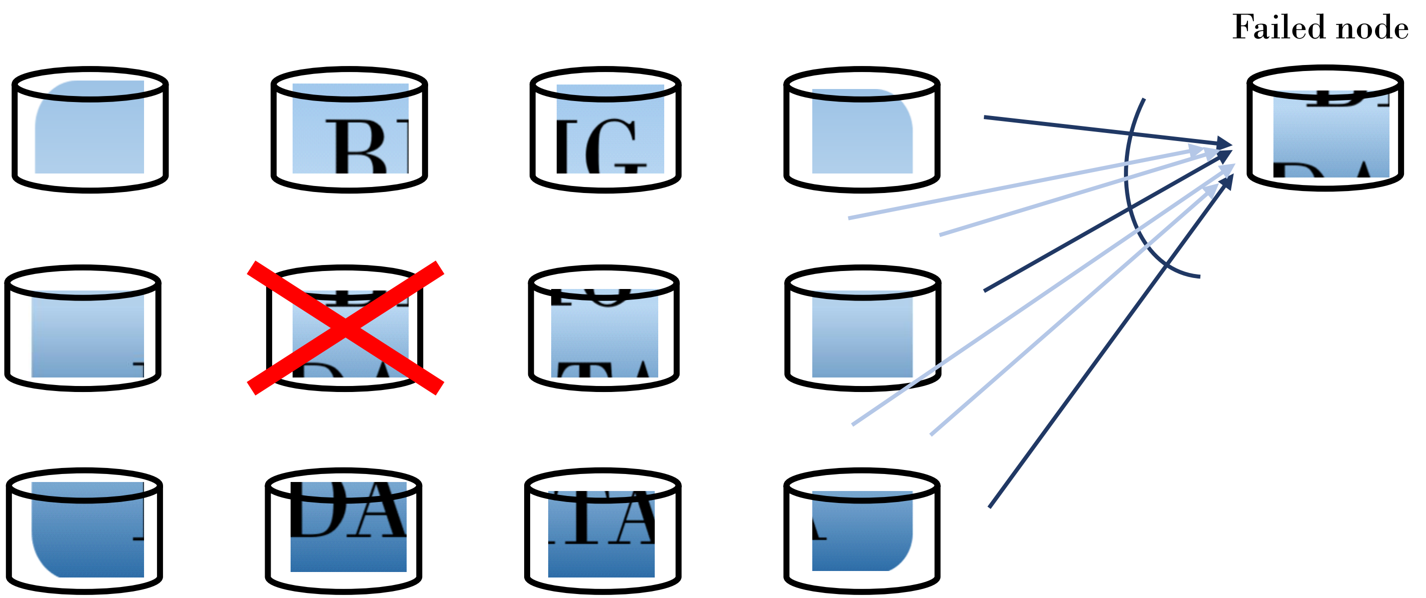
Contents

Locally Repairable Code

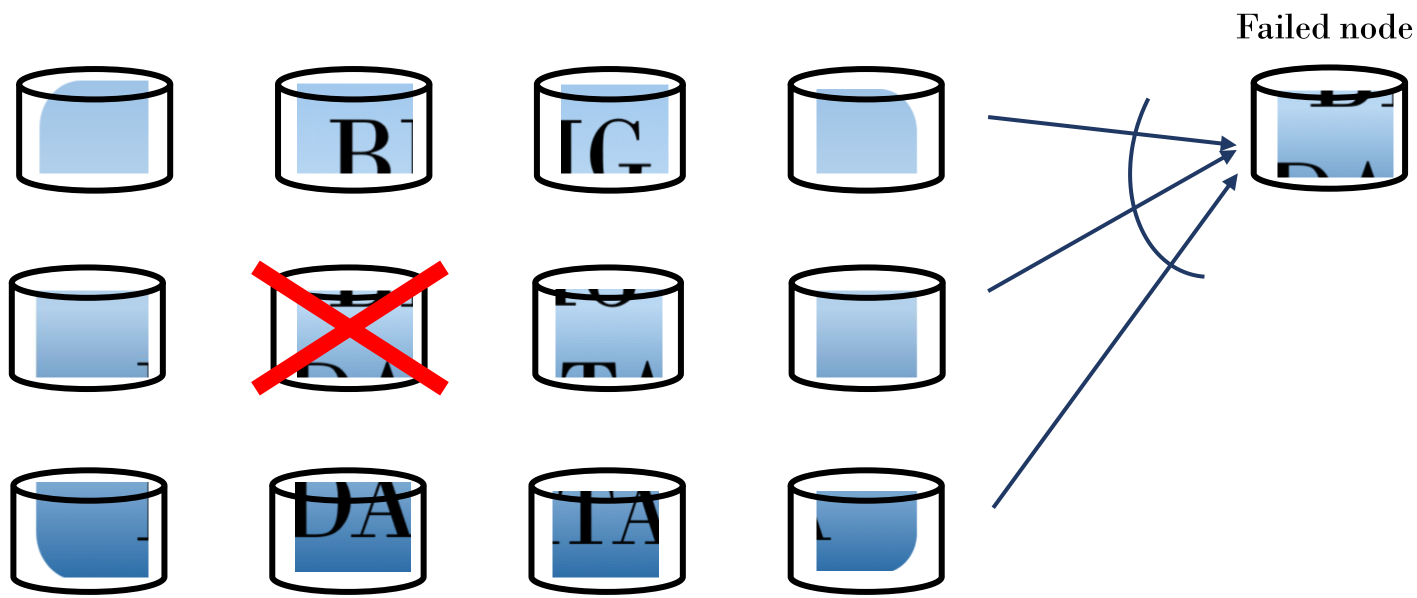
- To guarantee the reliability against node failures, various coding techniques have been applied



Locally Repairable Code

[Gopalan et al. 12]

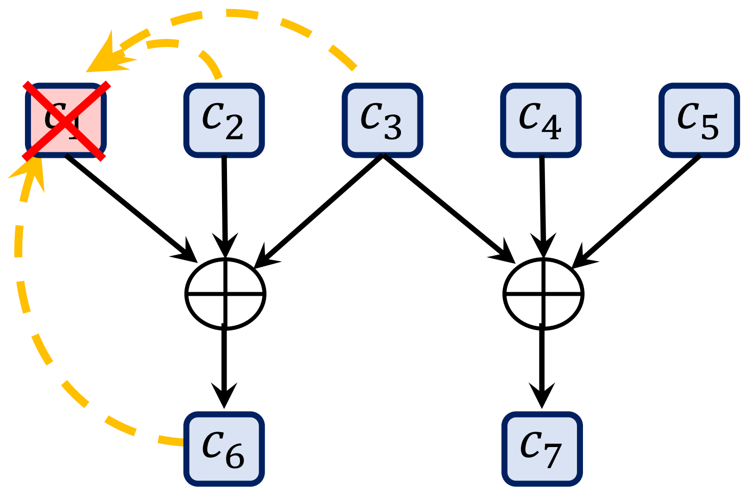
- Locally repairable code (LRC) just needs a small number of nodes to repair the single node failure



Locally Repairable Code

- **Locality**

  The number of nodes accessed to repair a single node failure



Locality of $c_1 \Rightarrow 3$

- Code C has locality r:

  All coded symbols have the locality at most r.

  C is denoted as $[n,k,r]$ LRC.

Locally Repairable Code

- **Availability**

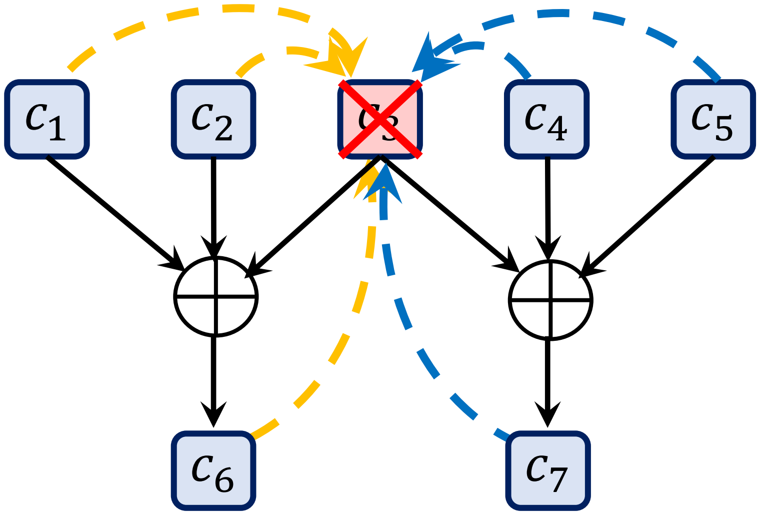
  The number of disjoint repair sets to repair a single node failure

Repair set of $c_3 \Rightarrow \{c_1, c_2, c_6\}$

Repair set of $c_3 \Rightarrow \{c_4, c_5, c_7\}$

Locality of $c_3 \Rightarrow 3$

Availability of $c_3 \Rightarrow 2$



LRCs for multiple erasure

- ***t*-parallel-recovery LRCs**
  - The repaired erasure **cannot** participate in the repair process of the unrepaired erasures

E.g. Erasures: $1^{st}, 2^{nd}$ symbol

Repaired locally and parallelly

i^{th} symbol	Repair set
1	{2,3} and {4,7}
2	{1,3} and {5,8}

- ***t*-sequential-recovery (t-seq) LRCs**
  - The repaired erasure **can** participate in the repair process of the unrepaired erasures

E.g. Erasures: $1^{st}, 2^{nd}, 7^{th}$ symbol

Locally repaired by order $7 \rightarrow 2 \rightarrow 1$

$2 \rightarrow 7 \rightarrow 1$

$2\ and\ 7 \rightarrow 1$

i^{th} symbol	Repair set
1	{2,3} and {4,7}
2	{1,3} and {5,8}
7	{1,4} and {8,9}

Disjoint repair group

- $[n] = \{1,2,\dots,n\}$
- h_i : i^{th} row of H

➢ Each row h_i defines a repair group since $h_i \cdot c = 0$ for any code word c.

$$H = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

repair group $\{1,2,3\}$ repair group $\{4,5,6\}$ repair group $\{7,8,9\}$

pairwise disjoint
and
$\{1,2,3\} \cup \{4,5,6\} \cup \{7,8,9\} = [9]$

If all these repair groups are **pairwise disjoint** and **their union becomes [n]**, then the LRC is said to have **disjoint repair groups**.

Disjoint repair group

- [9, 6, 2] LRC with disjoint repair group

$$H = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

repair group {1,2,3} repair group {4,5,6} repair group {7,8,9}

disjoint repair groups of all equal size r+1

⇒ length n is a multiple of r+1

$\Rightarrow m \triangleq \frac{n}{r+1}$

⇒ H matrix of size $m \times (r+1)m$

1. Preliminary

1) Locally repairable code (LRC)

2) LRCs for multiple erasure

3) Disjoint repair group

2. Intersections of two Binary LRCs with Disjoint Repair Group

1) necessary and sufficient condition for such an intersection to have availability 2

2) 3-seq LRCs with availability 2 constructed by such intersection

3) The dimension bound for 3-seq LRCs with availability 2

3. Concluding Remark

Intersections of Binary LRCs with Disjoint Repair Groups

1) necessary and sufficient condition for an intersection of LRCs to have availability 2

Theorem 1. Given two parity-check matrices H_1 and H_2 of the same size $m \times (r+1)m$ for two LRC's with disjoint repair groups and constant repair group size $r+1$ and with $m \geq r + 1$, the linear code C (which is the intersection of two constituent codes) with the parity check matrix

$$H = \begin{pmatrix} H_1 \\ \hdashline H_2 \end{pmatrix}$$

of size $2m \times (r+1)m$ will have availability 2 if and only if

$$\left|supp(h_{1,i}) \cap supp(h_{2,j})\right| = 1, \quad \text{for all } i, j,$$

where $h_{1,i}$ and $h_{2,j}$ are i-th row of H_1 and j-th row of H_2.

The proof is omitted.

2) 3-seq LRCs with availability 2 constructed by such intersection

Corollary 1. Assume the notation in Theorem 1. If the LRC in Theorem 1 with the parity check matrix H has availability 2, then this code is a 3-seq LRC.

Example)

$$c = (\not{c_1} \quad c_2 \quad c_3 \quad \not{c_4} \quad \not{c_5} \quad c_6 \quad c_7 \quad c_8 \quad c_9)$$

$$H = \begin{bmatrix} H_1 \\ \hdashline H_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ \hdashline 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$R_1(1) = \{2,3\}$, $R_2(1) = \{4,7\}$,

$R_1(4) = \{5,6\}$, $R_2(4) = \{1,7\}$,

$R_1(5) = \{4,6\}$, $R_2(5) = \{2,8\}$.

Locally repaired by order $c_1 \rightarrow c_4 \rightarrow c_5$ (or c_4 and c_5 in parallel)

or $c_5 \rightarrow c_1 \rightarrow c_4$ (or c_1 and c_4 in parallel)

⋮

The bound for 3-seq LRCs with availability 2

3) The dimension bound for 3-seq LRCs with availability 2

Theorem 2. Assume the notation in Theorem 1. If the LRC in Theorem 1 with the parity check matrix H has availability 2, then

$$rank(H) \geq 2m - \left\lfloor \frac{m}{r+1} \right\rfloor.$$

Corollary 2. If a linear code with the parity check matrix H of Theorem 1 has availability 2, then its dimension k is upper bounded by

$$k \leq (r-1)m + \left\lfloor \frac{m}{r+1} \right\rfloor \tag{3}$$

so that

$$\frac{k}{n} \leq \left(\frac{r-1}{r+1}\right) + \frac{1}{n}\left\lfloor \frac{n}{(r+1)^2} \right\rfloor \leq \frac{r^2}{(r+1)^2} \tag{4}$$

where the second inequality becomes equality if and only if $(r+1)|m$ or $(r+1)^2|n$.

The bound for 3-seq LRCs with availability 2

Remark 1.

$$\frac{k}{n} \leq \left(\frac{r-1}{r+1}\right) + \frac{1}{n}\left\lfloor\frac{n}{(r+1)^2}\right\rfloor \leq \frac{r^2}{(r+1)^2} \qquad (4)$$

the special case of H satisfying the condition in Theorem 1 for availability 2

→ rate optimal in the narrow sense of Theorem 1

general case [17]

- When $r+1$ does not divide m, this maximum rate is slightly smaller than $\frac{r^2}{(r+1)^2}$ in [17].

[17] W. Song, K. Cai, C. Yuen, K. Cai and G. Han, "On Sequential Locally Repairable Codes," IEEE Trans. Inf. Theory, vol. 64, no. 5, pp. 3513-3527, May 2018.

The bound for 3-seq LRCs with availability 2

Example 1 in [25]

$$H = \begin{pmatrix} I_4 & I_4 & I_4 \\ \hdashline I_4 & I_4^{(1)} & I_4^{(2)} \end{pmatrix}$$

The linear code with this H matrix is an $(12, 5, 2)$-LRC with availability 2, and in fact, it is a 3-seq LRC according to Cor. 1.

⇒ optimal in the sense of **Cor.2** :

$$k \le (r-1)m + \left\lfloor \frac{m}{r+1} \right\rfloor = 4 + \left\lfloor \frac{4}{3} \right\rfloor = 5$$

$$\frac{k}{n} = \frac{5}{12} \quad = \quad \left(\frac{r-1}{r+1}\right) + \frac{1}{n}\left\lfloor \frac{n}{(r+1)^2} \right\rfloor = \frac{1}{3} + \frac{1}{12}\left\lfloor \frac{4}{3} \right\rfloor = \frac{5}{12}$$

"**rate optimal** in the narrow sense of Theorem 1"

⇒ not optimal in the sense of **general rate bound** [17]:

$$\frac{k}{n} = \frac{5}{12} = 0.41\dot{6} \quad \le \quad \frac{r^2}{(r+1)^2} = \frac{4}{9} = 0.\dot{4}$$

[17] W. Song, K. Cai, C. Yuen, K. Cai and G. Han, "On Sequential Locally Repairable Codes," IEEE Trans. Inf. Theory, vol. 64, no. 5, pp. 3513-3527, May 2018.
[25] A. Wang and Z. Zhang and M. Liu, "Achieving arbitrary locality and availability in binary codes," in Proc. IEEE Int. Symp. Inf. Theory (ISIT), pp. 1866–1870, Jun. 2015.

The bound for 3-seq LRCs with availability 2

Example 8 in [27]

$$H = \begin{pmatrix} I_7 & I_7 & I_7 \\ \hdashline I_7 & I_7^{(1)} & I_7^{(3)} \end{pmatrix}$$

The linear code with this H matrix is an $(21, 8, 2)$-LRC with availability 2, and in fact, it is a 3-seq LRC according to Cor. 1.

⇒ **not optimal** in the sense of **Cor.2** :

$$k = 8 \quad \leq \quad (r-1)m + \left\lfloor \frac{m}{r+1} \right\rfloor = 7 + \left\lfloor \frac{7}{3} \right\rfloor = 9$$

Neither rate optimal in the sense of the **general rate bound [17]**, nor rate optimal in the narrow sense of **Cor.2**.

[27]E. Yavari and M. Esmaeili, "Locally Repairable Codes: Joint Sequential–Parallel Repair for Multiple Node Failures," IEEE Trans. Inf.Theory, vol. 66, no. 1, pp. 222-232, Jan. 2020.

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3. Concluding Remark

Concluding Remark

In this paper,

- Prove that a necessary and sufficient condition for intersections of two binary LRCs with disjoint repair group to have availability 2.

- Prove that such an intersection with availability 2 is in fact a 3-seq LRC.

- Prove that a bound on the dimension of such intersections.

- Find a relation between this bound and the general rate bound for 3-seq LRCs.

Thank you for listening