Trellis Structure and Performance Analysis of Space-Time Trellis Codes from Optimal Product Distance Codes

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Nov. 2. 2001

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- Trellis Structure of Opt. PDC

Performance Simulation

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**MIMO System with STTC**

### Block Diagram

![Block Diagram](image)

### Received Signal

\[
\begin{align*}
    r_t^j &= \sqrt{E_s / n} \sum_{i=1}^{n} \alpha_{j,i} c_i^j + \eta_t^j \\
    r_t &= \sqrt{E_s / n} \mathbf{H} \cdot \mathbf{c}_t + \mathbf{n}_t
\end{align*}
\]

### ML Decoding (Space-Time Viterbi Decoding)

\[
\begin{align*}
    \sum_{t=1}^{l} \sum_{j=1}^{m} \left| r_t^j - \sqrt{E_s / n} \sum_{i=1}^{n} \alpha_{j,i} c_i^j \right|^2 \\
    \tilde{\mathbf{C}} = \arg \min_{\mathbf{c}} \sum_{t=1}^{l} \left\| r_t - \sqrt{E_s / n} \mathbf{H} \cdot \mathbf{c}_t \right\|
\end{align*}
\]
Design Criteria for STTC

❖ Fundamental Bound (Tarokh et. al)

$$\Pr(c \rightarrow e) \leq \left( \frac{\eta E_s}{4N_0} \right)^{-rL_r}$$

❖ Rank Criterion

- Maximize the diversity advantage
  $$r = \text{rank}(f(c) - f(e))$$
  over all pairs of distinct codewords $c, e \in C$

❖ Determinant Criterion

- Maximize the coding advantage
  $$\eta = \left( \lambda_1 \lambda_2 \cdots \lambda_r \right)^{1/r}$$
  over all pairs of distinct codewords $c, e \in C$
  Where $\eta$ is the geometric mean of nonzero eigenvalues of
  $$A = (f(c) - f(e))(f(c) - f(e))^H$$
Tarokh et al
- Geometrically Uniform Code
  - Reduced complexity in computing the coding gain
- 4PSK: 4, 8, 16, 32 state, 8PSK: 8, 16, 32 state

Search Optimal STTC
- Based on Exhaustive Search
  - Baro: With Generator Matrix (2Tx, QPSK, 4, 8, 16 states)
  - Grimm: Within Zero-Symmetry Domain
  - Blum: With Coding gain Algorithm
- Mostly on the case of the number of Tx Antenna is 2
- BPSK, QPSK

Systematic Construction
- Hammons: From optimum $d_{\text{free}}$ convolutional code
- BPSK only
Delay Diversity = Repetition code + Delay element between multiple Tx Antennas
STTC using Opt. PDC

Transmitted Signal with Opt. PDC

Information Source → Optimum Product Distance code → Constellation Mapper → Signal N

D → Constellation Mapper → Signal N-1

D^{N-1} → Constellation Mapper → Signal 1
Consider a block Code $C$

$$C = \{c_1, c_2, \cdots, c_M\}$$

- $M$ codewords (with length $N$)
- $i$-th codeword

$$c_i = c_i^1 c_i^2 \cdots c_i^N, \quad c_i^m \in \mathbb{Z}_M$$

Product Distance

$$D(c_i, c_j) = \prod_{m=1}^{N} \left| f(c_i^m) - f(c_j^m) \right|$$

- With M-ary Modulation
- $f$ maps the symbol element to signal constellation
Product Distance Code

Example (QPSK, Tx Ant.: 3)

- Opt. Product Distance Code \{000, 112, 231, 323\}

Delay Diversity
\{000, 111, 222, 333\}
- Minimum Dist. = 8
- # of Minimum = 4
- Avg. Dist. = 26.67

Opt. PDC
\{000, 112, 231, 323\}
- Minimum Dist. = 16
- # of Minimum = 6
- Avg. Dist. = 16.00

Diagram:
- Nodes: 0, 1, 2, 3
- Edges:
  - 0 to 2 with distance \((\sqrt{2})^2\)
  - 2 to 4 with distance \(2^2\)
STTC using Opt. PDC

FER Performance [QPSK, 2Tx, 4State, 2b/s/Hz]

Frame Error Rate (FER)

SNR per Receive Antenna [dB]

- Opt.PDC\text{ A} (1Rx)
- Opt.PDC\text{ B} (1Rx)
- Opt.PDC\text{ A} (2Rx)
- Opt.PDC\text{ B} (2Rx)
- Opt.PDC\text{ A} (4Rx)
- Opt.PDC\text{ B} (4Rx)
Example (QPSK, Tx Ant.: 2)

<table>
<thead>
<tr>
<th>Opt. PDC A</th>
<th>Opt. PDC B</th>
</tr>
</thead>
<tbody>
<tr>
<td>{00, 11, 22, 33}</td>
<td>{00, 11, 23, 32}</td>
</tr>
<tr>
<td>Minimum Dist. = 4</td>
<td>Minimum Dist. = 4</td>
</tr>
<tr>
<td># of Minimum = 4</td>
<td># of Minimum = 2</td>
</tr>
<tr>
<td>Avg. Dist. = 8</td>
<td>Avg. Dist. = 6.67</td>
</tr>
</tbody>
</table>

Minimum Dist. = 4
# of Minimum = 4
Avg. Dist. = 8

Minimum Dist. = 4
# of Minimum = 2
Avg. Dist. = 6.67
Design Criterion

 Definitions

- To Maximize the coding gain

\[ D_{\text{min}} = \min_{i \neq j} D_{(c_i, c_j)} = \min_{i \neq j} \prod_{m=1}^{N} \left| f(c_i^m) - f(c_j^m) \right| \]

- \( N_{\text{min}} \): the number of distinct codeword pairs \((c_i, c_j)\) with

\[ D_{(c_i, c_j)} = D_{\text{min}} \]

- \( D_{\text{avg}} \): the average product distance between pairs of distinct codewords

 Criterion

- Opt. PDC with \( N_{\text{min}} \) minimized
- Opt. PDC with \( D_{\text{avg}} \) maximized
### Search Results (QPSK Opt. PDC)

#### QPSK

<table>
<thead>
<tr>
<th>Tx</th>
<th>PDC</th>
<th>$D_{opt}$</th>
<th>$N_{min}$</th>
<th>$D_{avg}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0 1 2 3</td>
<td>0 1 3 2</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>0 1 2 3</td>
<td>0 1 2 3</td>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>0 1 2 3</td>
<td>0 1 3 2</td>
<td>16</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>0 2 1 3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0 1 2 3</td>
<td>0 1 2 3</td>
<td>32</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>0 1 3 2</td>
<td>0 2 1 3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0 1 2 3</td>
<td>0 1 3 2</td>
<td>64</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>0 2 1 3</td>
<td></td>
<td></td>
<td>4</td>
</tr>
</tbody>
</table>
## Search Results (8PSK, 16QAM)

### 8PSK

<table>
<thead>
<tr>
<th>Tx</th>
<th>PDC</th>
<th>$D_{opt}$</th>
<th>$N_{min}$</th>
<th>$D_{avg}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0 1 2 3 4 5 6 7</td>
<td>2</td>
<td>16</td>
<td>4.57143</td>
</tr>
<tr>
<td></td>
<td>0 3 6 1 4 7 2 5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0 1 2 3 4 5 6 7</td>
<td>4</td>
<td>12</td>
<td>6.28571</td>
</tr>
<tr>
<td></td>
<td>0 3 7 4 1 6 2 5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0 2 4 7 1 6 3 5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0 3 6 2 5 7 1 4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0 1 2 3 4 5 6 7</td>
<td>4</td>
<td>2</td>
<td>25.5714</td>
</tr>
<tr>
<td></td>
<td>0 1 5 3 7 2 6 4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0 3 2 5 1 7 6 4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0 3 7 5 4 2 6 1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0 1 2 3 4 5 6 7</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0 3 7 4 1 6 2 5</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### 16QAM

<table>
<thead>
<tr>
<th>Tx</th>
<th>PDC</th>
<th>$D_{opt}$</th>
<th>$N_{min}$</th>
<th>$D_{avg}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0 1 2 3 4 5 6 7</td>
<td>64</td>
<td>4</td>
<td>426.667</td>
</tr>
<tr>
<td></td>
<td>0 6 13 11 9 15 4 2 7 11 0 12 14 8 3 5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0 1 2 3 4 5 6 7</td>
<td>33</td>
<td>33</td>
<td>494.933</td>
</tr>
<tr>
<td></td>
<td>0 2 4 6 8 10 12 14 1 3 5 7 9 11 13 15</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
**Trellis Structure of Opt. PDC**

Given Opt. PDC (M-ary, n Tx Ant)

\[
\begin{bmatrix}
0 & 0 & \cdots & 0 \\
1 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
M-1 & M-1 & \cdots & M-1
\end{bmatrix}
\]

\[
\begin{bmatrix}
c_0^1 & c_0^2 & \cdots & c_0^N \\
c_1^1 & c_1^2 & \cdots & c_1^N \\
\vdots & \vdots & \ddots & \vdots \\
c_{M-1}^1 & c_{M-1}^2 & \cdots & c_{M-1}^N
\end{bmatrix}
\]

**Trellis Structure**

Current Input: \( I \)

\[
f(c_{S_1}^N) \\
f(c_{S_2}^{N-1}) \\
\vdots \\
f(c_{S_{N-1}}^2) \\
f(c_I^1)
\]

Transition Output

Next state

\[
\begin{bmatrix}
S_1 & S_2 & S_3 & \cdots & S_{N-1} \\
S_2 & S_3 & \cdots & I
\end{bmatrix}
\]
System Description

Channel Model
- Quasi-static flat fading channel + AWGN Channel
- 1 frame = 130 symbols (l=130, IS-54)

\[ r = Hc + n \quad H = [\alpha_{ij}] = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{bmatrix} \]

- Ideal channel estimation

System model
- Modulation: QPSK, 8PSK, 16QAM
- Space-Time Viterbi Decoder (ML)
- Unquantized Soft Decision
Delay VS Opt. PDC [QPSK, 2Tx]

FER Performance [QPSK, 2Tx, 4State, 2b/s/Hz]

Frame Error Rate (FER) vs. SNR per Receive Antenna [dB]

- Delay (1Rx)
- Pd_{opt} (1Rx)
- Delay (2Rx)
- Pd_{opt} (2Rx)
- Delay (4Rx)
- Pd_{opt} (4Rx)
Delay VS Opt. PDC [QPSK, 3Tx]

FER Performance [QPSK, 3Tx, 16State, 2b/s/Hz]

Frame Error Rate (FER) vs. SNR per Receive Antenna [dB]

- Delay (1Rx)
- Delay (2Rx)
- Delay (4Rx)
- Pd_{opt} (1Rx)
- Pd_{opt} (2Rx)
- Pd_{opt} (4Rx)
FER Performance [QPSK, 4Tx, 64State, 2b/s/Hz]

SNR per Receive Antenna [dB]

Frame Error Rate (FER)

Delay(1Rx)
Pd_{opt} (1Rx)
Delay(2Rx)
Pd_{opt} (2Rx)
Delay(4Rx)
Pd_{opt} (4Rx)
Delay VS Opt. PDC [8PSK, 2Tx]

FER Performance [8PSK, 2Tx, 8State, 3b/s/Hz]

Frame Error Rate (FER)

SNR per Receive Antenna [dB]
Delay VS Opt. PDC [8PSK, 3Tx]

FER Performance [8PSK, 3Tx, 64State, 3b/s/Hz]
Delay VS Opt. PDC [16QAM, 2Tx]

FER Performance [16QAM, 2Tx, 16State, 4b/s/Hz]

Frame Error Rate (FER) vs. SNR per Receive Antenna [dB]

Graph showing the Frame Error Rate (FER) performance for different delay and Pdt.N values with 1, 2, and 4 receive antennas.
Concluding Remarks

- STTC from Opt. PDC
  - From the Optimum Product Distance Codes
  - Modified Design Criterion by Product Distance Profile
  - Applicable to the case of the number of Tx Ant. is 2,3,4…

- Search Optimal Product Distance Code
  - 4PSK, 8PSK, 16QAM
  - Set up Trellis structure from Opt. PDC

- Performance Simulation
  - Comparison with Delay Diversity (Baseline Performance)