Two-dimensional Patterns with Optimal Auto- and Cross-Correlation Functions

Hong Y. Song and Solomon W. Golomb[1]

Comm. Sci. Inst., 3740 McClintock Ave., EEB-500, Univ. of Southern Calif., Los Angeles, CA 90089-2565

In this note, we consider a set of sonar arrays of the same size with an additional constraint that the non-periodic two-dimensional cross-correlation value for any two distinct arrays is limited to either 1 or 0. These find applications in various multiuser communication systems such as multiuser radar and sonar systems [1] and/or fiberoptic CDMA networks. [2]

We define an *optimal* pair of sonar arrays to be a pair of sonar arrays having an ideal cross-correlation function and having the maximum possible number of columns for a fixed number of rows. We first prove that an optimal pair having n rows can have at most n columns. It took only about 8 min. of CPU time in a Sun Sparc Station to find the following pair of 8×8 sonar arrays, but it spent more than 100 hours for the size 9×9 without finding any example. Here, $f_A(j) = i$ implies the array A has a dot in row i and column j.

j	1	2	3	4	5	6	7	8
$f_A(j)$	1	3	4	8	2	7	2	8
$f_B(j)$	6	4	1	8	8	1	4	3

More generally, a set S of k sonar arrays of size $kn \times n$ having ideal cross-correlation functions can be constructed from "(n, k)-sequences of length kn." [3] Whenever $kn + 1 = p$ is an odd prime, the construction can best be described as follows: Let α be a primitive root mod p. Then, each $A_j \in$ S for $j = 0, 1, 2, \ldots, k-1$ has a dot in row α^{ki+j} for every column $i = 1, 2, \ldots, n$. When $k = 1$, this produces the Welch-Costas array of order $p - 1$ [4, 5], which has an ideal two-dimensional non-periodic autocorrelation function. Note that no two Costas arrays of the same size can have an ideal cross-correlation function. [6, 7]

When $k = 2$, the above construction produces a pair of sonar arrays of size $2n \times n$ having an ideal non-periodic two-dimensional cross-correlation function, by taking even-numbered columns for one array and taking odd-numbered columns for the other from the Welch-Costas array. Note that adjoining one empty row at the bottom of the Welch-Costas array results in a "doubly-periodic" sonar array. Therefore, a similar decomposition of this array gives a pair of "doubly-periodic" $p \times (p-1)/2$ sonar arrays having an ideal cross-correlation function. On the other hand, we will show that an optimal pair of "doubly-periodic" sonar arrays having $2n + 1$ rows can have at most n columns. Therefore, the above construction produces an optimal pair of doubly-periodic sonar arrays.

A similar technique as in [8] can be applied to find a better pair of known sonar arrays having an ideal cross-correlation function. Let α be a primitive root mod $p = 2n + 1$ (an odd prime), and let $f(i) = \alpha^i$ where $i = 1, 2, \ldots, p-1$ denote a $p \times (p-1)$ "modular sonar array." For any integers a, b, and c, $g(i) = af(i) + bi + c \pmod p$ is also a $p \times (p-1)$ modular sonar array. [8] We finally show that for any integers a, b, c_A, c_B, two arrays given by $g_A(i) = af(2i) + b(2i) + c_A$ and $g_B(i) = af(2i-1) + b(2i-1) + c_B \pmod p$ for $i = 1, 2, \ldots, (p-1)/2$ are sonar arrays having an ideal cross-correlation function. For an appropriate choice of parameters (a, b, c_A, c_B), the resulting arrays will have longer successive empty rows at the bottom to be deleted. The following table shows the computer results for p up to 97.

p	size	p	size
7	3×3	47	35×23
11	6×5	53	40×26
13	7×6	59	48×29
17	10×8	61	49×30
19	11×9	67	54×33
23	14×11	71	59×35
29	21×14	73	60×36
31	22×15	79	65×39
37	27×18	83	69×41
41	30×20	89	76×44
43	31×21	97	83×48

References

[1] S. V. Maric and E. L. Titlebaum, "A class of frequency hop codes with nearly ideal characteristics for use in multiple-access spread-spectrum communications and radar and sonar systems," *IEEE Trans. Communications*, vol. 40, no. 9, pp. 1442-1447, Sept., 1992.

[2] R. Gagliardi and H. Taylor, "Code sequences and code matrices for CDMA fiberoptic systems," Tech. Report CSI-91-08-01, Comm. Sci. Inst., Univ. of Southern Calif., 1991.

[3] Hong Y. Song and S. W. Golomb, "Generalized Welch-Costas sequences and their application to Vatican arrays," *Contemporary Mathematics*, to appear.

[4] J. P. Costas, "A study of a class of detection waveforms having nearly ideal range-doppler ambiguity properties," *Proceedings of the IEEE*, vol. 72, no. 8, pp. 996-1009, Aug., 1984.

[5] S. W. Golomb and H. Taylor, "Constructions and properties of Costas arrays," *Proceedings of the IEEE*, vol. 72, no. 9, pp. 1143-1163, Sept. 1984.

[6] Herbert Taylor, "Non-attacking rooks with distinct differences," Tech. Report CSI-84-03-02, Comm. Sci. Inst., Univ. of Southern Calif., 1984.

[7] A. Freedman and N. Levanon, "Any two $N \times N$ Costas signals must have at least one common ambiguity sidelobe if $N > 3$ — A proof," *Proceedings of the IEEE*, vol. 73, no. 10, pp. 1530-1531, Oct. 1985.

[8] O. Moreno, R. Games, and H. Taylor, "Sonar sequences from Costas arrays and the best known sonar sequences with up to 100 symbols," *IEEE Trans. on Info. Theory*, vol. 39, no. 6, pp. 1985-1987, 1993.

[1] This work was supported in part by the United States Office of Naval Research under Grant No. N00014-90-J-1341.