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# Quasi-Hadamard Matrix

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# Hadamard Matrix

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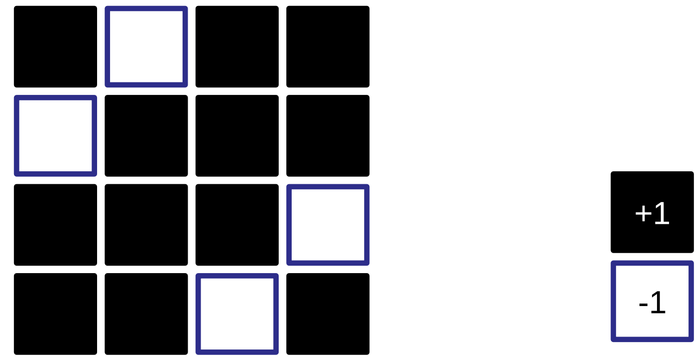
- A Hadamard matrix of order $n$ (or, size $n \times n$) is defined as an $n \times n$ matrix with all entries +1 or -1 such that

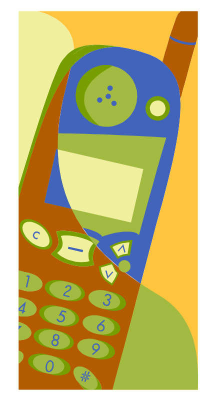
$$H H^T = n I,$$

where $I$ is the $n \times n$ identity matrix.




- **Hadamard Conjecture:** There exists a Hadamard matrix of order every multiple of 4.

- **Hadamard matrix is widely used in communications and signal processing engineering:**
  - Orthogonal channelization in CDMA communications
  - Construction of orthogonal signals
  - Construction of GOOD error-correcting codes

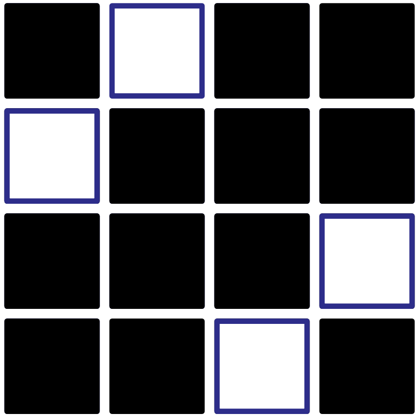


# Hadamard Equivalence

**Definition 1 (Hadamard Equivalence)**

Two **binary matrices** of the same size are said to be hadamard-equivalent (or just **equivalent**) if one can be converted to the other by some combinations of the following hadamard-preserving operations:

- CC/CR: Complementing a column (CC) / a row (CR)
- PC/PR: Permuting columns (PC) / rows (PR)



| Size | # inequivalent Hadamard matrices | Reference |
|---|---|---|
| 1, 2, 4, 8, 12 | 1 | |
| 16 | 5 | |
| 20 | 3 | |
| 24 | 60 | Kimura, 1989 |
| 28 | 487 | Kimura, 1994 |
| ≥32 | Unknown | |

# Absolute correlation is preserved

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- Give two **binary vectors** $\underline{r}$ and $\underline{s}$ of length n, their absolute correlation is given as

$$C(\underline{r},\underline{s}) = \left| \sum_i (-1)^{r(i)+s(i)} \right| = |A - D|$$

where $A$ is the number of agreements and $D$ is the number of disagreements between $\underline{r}$ and $\underline{s}$.

**Remark 1.** The absolute correlation of the two rows of a 2 x n binary matrix will be preserved by any Hadamard-preserving operation.

**Proposition 1.** Two equivalent $m \times n$ **binary matrices** have the same profile of absolute correlations of the rows.

# Integer Representation of Binary Matrices

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**Definition 2**: Let $A = (a_{ij})$ be an $m \times n$ binary matrix where $i = 1, 2, \ldots, \mathrm{m}$ and $j = 1, 2, \ldots, \mathrm{n}$. We define a map $\rho$ as

$$\rho(A) \triangleq \sum_{i=1}^{m} \sum_{j=1}^{n} \left[ a_{ij} 2^{n(m-i)+(n-j)} \right]$$

**Example:**

$$\rho \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} = \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{matrix}$$

$$= 0000001101010110_{(2)} = 854.$$

- **Note that the map ρ is bijective**

**Definition 3.** The minimal matrix of an equivalence class is called the **Hadamard-row minimal matrix**, or **HR-minimal**. Its $\rho$ value is called the $\rho$ value of the equivalence class.

# Example 1: 2 x 2 binary matrices

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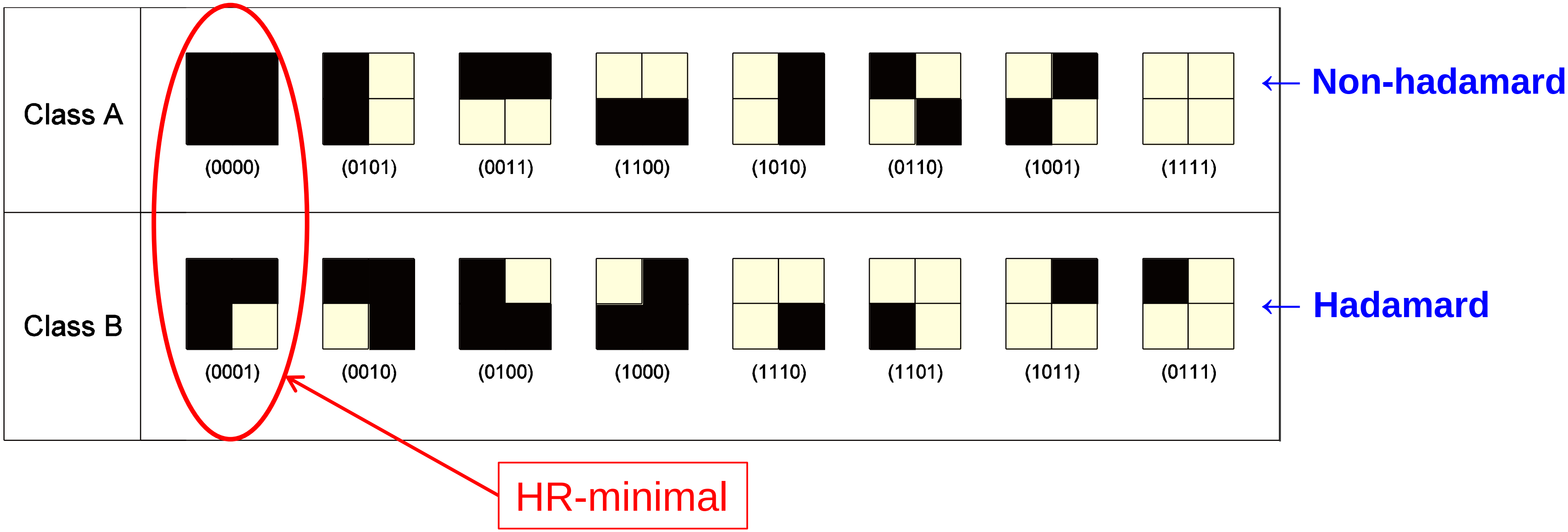

# Example 2:    some more

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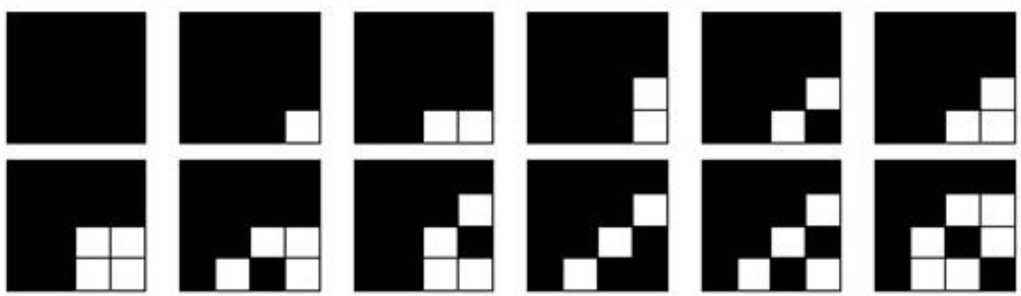
| Size | Number | Inequivalent HR-minimals | $\rho$ values |
|---|---|---|---|
| 2 x 2 | 2 |  | 0, **1** |
| 2 x 3 | 2 |  | 0, **1** |
| 2 x 4 | 3 |  | 0, 1, **3** |
| 3 x 3 | 3 |  | 0, 1, **10** |
| 3 x 4 | 5 |  | 0, 1, 3, 18, **53** |
| 4 x 4 | 12 |  | 0, 1, 3, 17, 18, 19<br>51, 52, 291, 292, 293, **854** |

# Shape/Properties of HR-minimals

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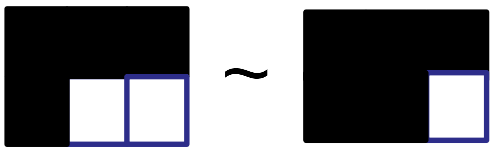
**Theorem 2.**

1) An HR-minimal is in a normalized form. That is, its top row and left-most column consist entirely of 0’s.
2) In an HR-minimal of size $m \times n$, then weight of the second row cannot exceed $n/2$. Furthermore, in the second row, all the 0’s come to the left of all the 1’s. In its second most column, all the 0’s come on top of all the 1’s.

   **Remark 1.** It seems to be true that the weight of the second column of an $m \times n$ HR-minimal cannot exceed $m/2$. (open)

3) An HR-minimal is row-sorted and column-sorted.

   **Remark 2.** Its converse is not true.



# Shape/Properties of HR-minimals

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**Corollary 1:** Two same rows of an HR-minimal must be adjacent. So must be two same columns.

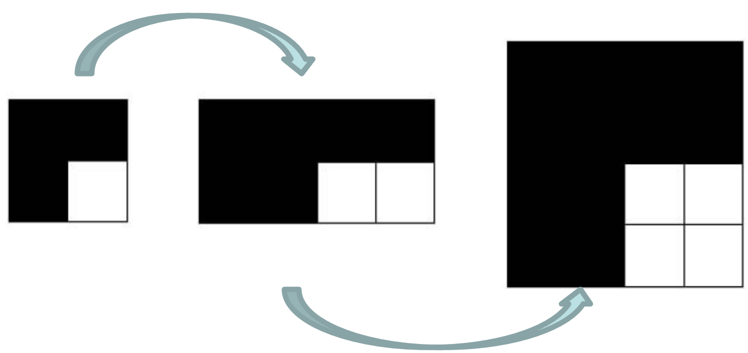
**Proposition 4:** In an HR-minimal, the number of row-repetitions of any row cannot exceed that of the all-zero row at the top.

**Remark 3**: Similar statement for the columns is **not true** in general.

**Theorem 2 (Linear Expanding Construction)**: Let $A = (a_{ij})$ be an $m \times n$ HR-minimal, and $k$ and $l$ be positive integers. Then $B = (b_{ij})$ of size $km \times ln$ is also an HR-minimal, where

$$b_{ij} = a_{\lfloor \frac{i+k-1}{k} \rfloor \lfloor \frac{j+l-1}{l} \rfloor}$$

**Corollary 2 (Add-zero-row):** We can construct an $(m+1) \times n$ HR-minimal by adjoining the all-zero-row at the top of an $m \times n$ HR-minimal.



**Remark 4:** Repeating any other row not necessarily preserves the HR-minimality.

# Shape/Properties of HR-minimals

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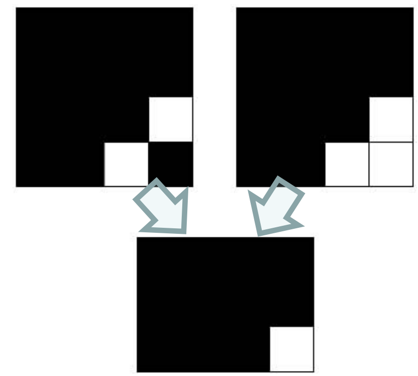
**Theorem 3 (Add-zero-column):** We can construct an $m\times(n+1)$ HR-minimal by adjoining the all-zero-column at the left-most of an $m\times n$ HR-minimal. (proved after ISIT2010 submission)

**Proposition 5:** If $A$ is an $m\times n$ HR-minimal, then the $(m\text{-}1)\times n$ matrix obtained by deleting the bottom row of $A$ is also an HR-minimal.



**Remark 5**. Deleting the right-most column of an HR-minimal does not in general result in an HR-minimal.

| | | | | | | |
|---|---|---|---|---|---|---|
| 000000 | | 00000 | | 00000 | | 00000 |
| 000011 | | 00001 | | 00001 | | 00001 |
| 001100 | → | 00110 | ~ | 01010 | ~ | 00110 |
| 010101 | | 01010 | | 00110 | | 00111 |
| 010110 | | 01011 | | 00111 | | 01010 |

# Weight of the second row of HR-minimal

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- If the weight of the second row is $w$, then the correlation of the top row (= all-zero-row) and the second row becomes:

$$\#\text{Agreements} - \#\text{Disagreements} = n - 2w.$$

**Proposition 6.** The value $|n - 2w| = C_{max}$ is the **maximum** (in absolute value) over the correlations of all possible pairs of rows of **an HR-minimal**.

- Therefore, the HR-minimal $A$ with largest weight in its second row gives a set of row vectors with the lowest possible pair-wise correlations.

**Definition 5 (Quasi-Hadamard Matrix)**

(a) An $m \times n$ equivalence class containing an HR-minimal $A$ is called a **Quasi-Hadamard class (QH class)** if the weight of the second row of $A$ satisfies the following:

① $w$ = ($n$-1)/2 when $n$ is odd;

② $w$ = $n$/2 when $n$ is doubly-even;

③ $w$ = $n$/2 -1 when $n$ is singly-even and $m$ > 2;

④ $w$ = $n$/2 when $n$ is singly-even and $m$ = 2;

(b) All the matrices in the equivalence class containing such $A$ are called **Quasi-Hadamard matrices (QH matrices).** The HR-minimal $A$ is called a **minimal QH matrix**.

(c) Define $R_Q(n)$ to be the **maximum** such that an $R_Q(n) \times n$ minimal QH matrix exists.

# Number of inequivalent $R_Q(n) \times n$ minimal QH matrices

| $n$ | $R_Q(n)$ | # | $n$ | $R_Q(n)$ | # |
|---|---|---|---|---|---|
| 3 | 4 | 1 | **12** | **12** | **1** |
| **4** | **4** | **1** | 13 | 13 | 1 |
| 5 | 5 | 1 | 14 | ≥ 16 | ≥ 1 at m=16 |
| 6 | 16 | 1 | 15 | 16 | 5 |
| 7 | 8 | 1 | **16** | **16** | **5** |
| **8** | **8** | **1** | 17 | 16 | 76 |
| 9 | 8 | 1 | 18 | ≥ 20 | ≥ 1 at m=20 |
| 10 | 16 | 3 | 19 | 20 | 3 |
| 11 | 12 | 1 | **20** | **20** | **3** |

# Some Examples of minimal QH Matrices

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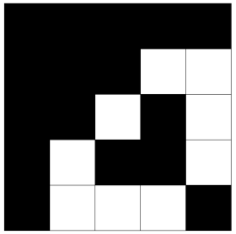
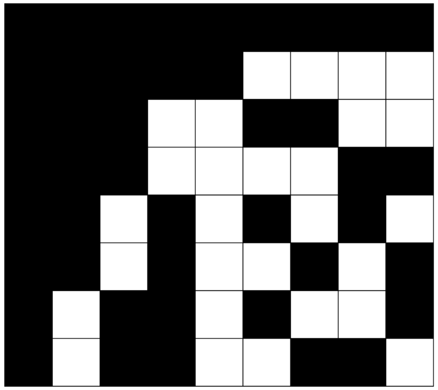
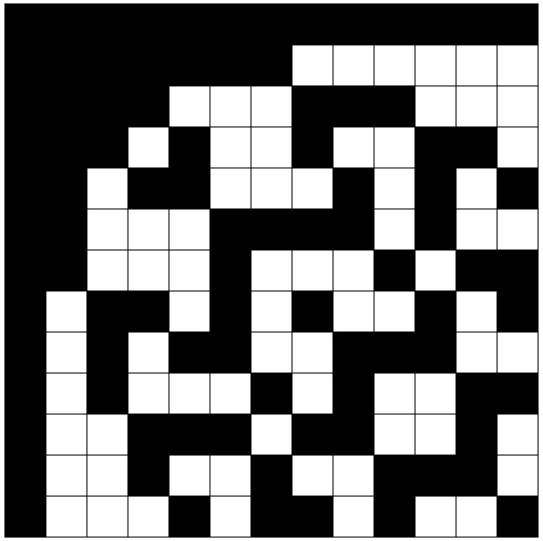


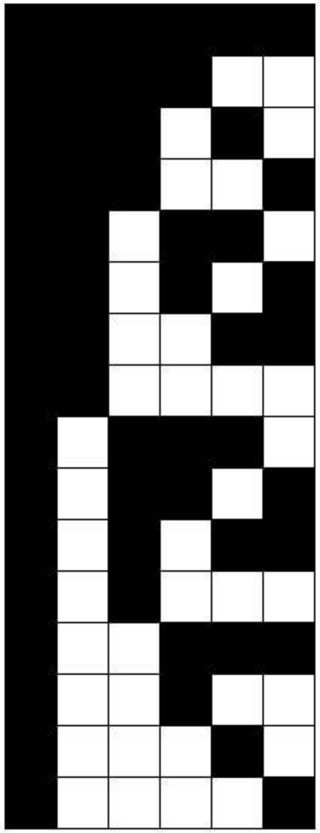
5x5



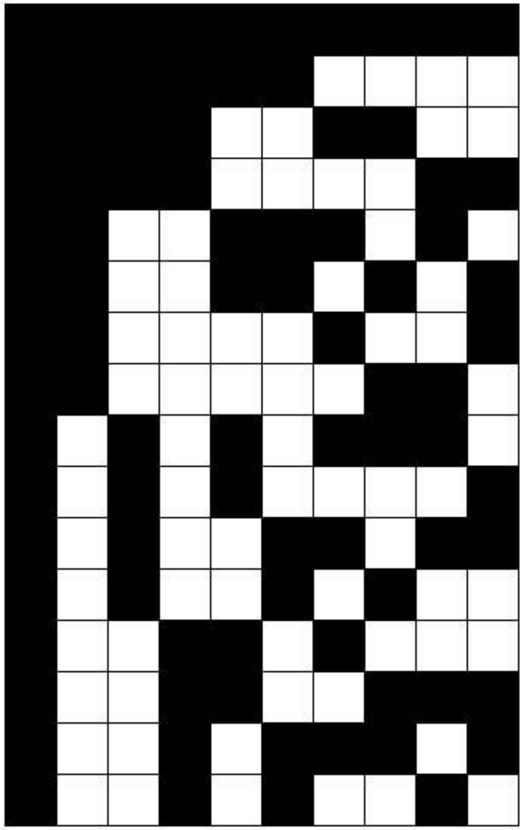
8x9



13x13



16x6



16x10

# Properties of QH matrices

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## Corollary 4.

1) Deleting the bottom row of an $m \times n$ minimal QH matrix gives an $(m\text{-}1) \times n$ minimal QH matrix.
2) There does not exist an $(m\text{-}1) \times n$ minimal QH matrix if these does not exist an $m \times n$ minimal QH matrix

**Remark 5**

1) For all $n \geq 2$, there exists a unique $2$x$n$ QH class.
2) For $2 < m \leq R_Q(n)$, the number of $m$x$n$ QH classes could be 1 or more.
3) An $m$x$n$ QH matrix is a Hadamard matrix when $m=n$ and $n=0$ (mod 4).

**Corollary 3.** An $m$x$n$ minimal QH matrix has a larger $\rho$ value than any other HR-minimals of non-QH classes.

**Remark 6.** When we consider all the HR-minimals of size $m$x$n$ and order them according to the $\rho$ values in increasing order, then all the minimal QH matrices come at the end.

**Conjecture 1:** There exists an $n \times n$ QH class for all $n \geq 2$ except for $n \equiv 1$ (mod 8).

**Remark 7:** The answer YES to this problem for **$n \equiv 0$ mod $4$** implies and implied by the Hadamard Conjecture.

# HC-minimal and H-minimal

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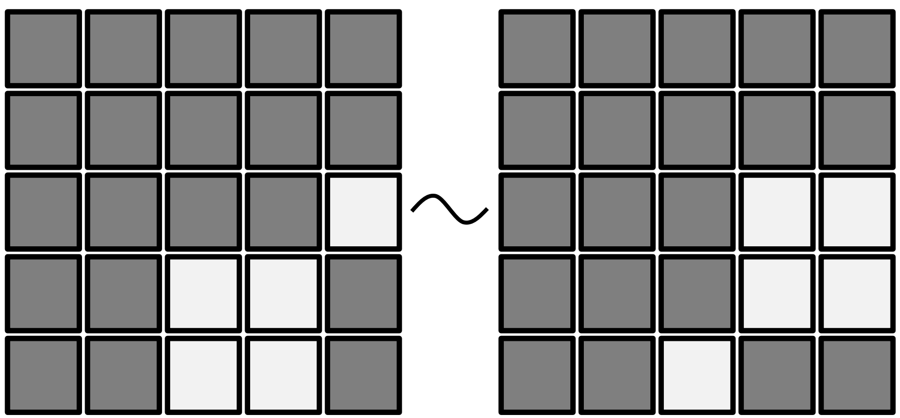
**Definition 4.** If a binary matrix $A^T$ is an HR-minimal (of some equivalence class), then we call $A$ a Hadamard-Column minimal matrix, or **HC-minimal**. If an HR-minimal is also an HC-minimal, then we call it **H-minimal**.

**Remark 7**. An HR-minimal is not always an HC-minimal. Thus, not every class contains an H-minimal.



HR-minimal      Not HR-minimal

# Number of Equivalence Classes

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- We denote by $N_E(m, n)$ the number of equivalence classes of binary matrices of the size $m \times n$.

**Proposition 9:**

- For a given size $m \times n$ , the number of HR-minimals is the same as that of HC-minimals.
- $N_E(m, n) = N_E(n, m)$ for any positive integers $m$ and $n$.

**Proposition 10:** $N_E(m, n)$ is monotonically non-decreasing as $m$ or $n$ increases.

# Some Formula

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## Theorem 4:

$$N_E(1,n) = 1$$

$$N_E(2,n) = \sum_{a=0}^{\lfloor \frac{n}{2} \rfloor} 1 = \left\lfloor \frac{n}{2} \right\rfloor + 1$$

$$N_E(3,n) = \sum_{a=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{b=\lfloor \frac{a}{2} \rfloor}^{\lfloor \frac{n-a}{2} \rfloor} \sum_{c=\max(0,a-b)}^{\lfloor \frac{a}{2} \rfloor} 1$$

| m \ n | 1 | 2 | 3 |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 2 | 1 | 2 | 2 |
| 3 | 1 | 2 | 3 |
| 4 | 1 | 3 | 5 |
| 5 | 1 | 3 | 6 |
| 6 | 1 | 4 | 9 |
| 7 | 1 | 4 | 11 |
| 8 | 1 | 5 | 15 |
| 9 | 1 | 5 | 18 |
| 10 | 1 | 6 | 23 |
| 11 | 1 | 6 | 27 |
| 12 | 1 | 7 | 34 |
| 13 | 1 | 7 | 39 |
| 14 | 1 | 8 | 47 |
| 15 | 1 | 8 | 54 |
| 16 | 1 | 9 | 64 |
| 17 | 1 | 9 | 72 |
| 18 | 1 | 10 | 84 |
| 19 | 1 | 10 | 94 |
| 20 | 1 | 11 | 108 |

# Number of the Equivalence Classes

| m \ n | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | | | | | | | |
| 2 | 1 | **2** | | | | | | |
| 3 | 1 | 2 | **3** | | | | | |
| 4 | 1 | 3 | 5 | **12** | | | | |
| 5 | 1 | 3 | 6 | 18 | **39** | | | |
| 6 | 1 | 4 | 9 | **35** | **101** | **388** | | |
| 7 | 1 | 4 | 11 | **54** | **228** | **1343** | **8102** | |
| 8 | 1 | 5 | 15 | **94** | **551** | **5083** | **53775** | **656108** |
| 9 | 1 | 5 | 18 | **140** | **1221** | **18366** | **355773** | **?** |
| 10 | 1 | 6 | 23 | **224** | **2746** | **66524** | **?** | **?** |
| 11 | 1 | 6 | 27 | **326** | **5850** | **231189** | **?** | **?** |
| 12 | 1 | 7 | 34 | **495** | **12338** | **780372** | **?** | |
| 13 | 1 | 7 | 39 | **699** | **24994** | **?** | **?** | |
| 14 | 1 | 8 | 47 | **1012** | **49708** | **?** | | |
| 15 | 1 | 8 | 54 | **1397** | **95771** | **?** | | |
| 16 | 1 | 9 | 64 | **1955** | **180759** | **?** | | |

# Number of H-minimals

| m \ n | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 1 | **1** | | | | | | | |
| 2 | 1 | **2** | | | | | | |
| 3 | 1 | 2 | **3** | | | | | |
| 4 | 1 | 3 | 5 | **12** | | | | |
| 5 | 1 | 3 | 6 | 18 | **37** | | | |
| 6 | 1 | 4 | 9 | **34** | **93** | **318** | | |
| 7 | 1 | 4 | 11 | **53** | **197** | **968** | **4624** | |
| 8 | 1 | 5 | 15 | **90** | **448** | **3109** | **23518** | **200127** |
| 9 | 1 | 5 | 18 | **131** | **917** | **9549** | **118346** | **?** |
| 10 | 1 | 6 | 23 | **205** | **1913** | **29244** | **?** | **?** |
| 11 | 1 | 6 | 27 | **292** | **3728** | **85549** | **?** | |
| 12 | 1 | 7 | 34 | **434** | **7285** | **?** | | |
| 13 | 1 | 7 | 39 | **?** | **?** | **?** | | |
| 14 | 1 | 8 | 47 | **?** | **?** | | | |
| 15 | 1 | 8 | 54 | | | | | |
| 16 | 1 | 9 | 64 | | | | | |

# Some Open Problems

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1. Find a systematic construction for HR-minimals or minimal QH matrices of size $m$ x $n$ *(except for using cyclic difference sets)*
2. The weight of the second column of an $m$ x $n$ HR-minimal cannot exceed $m/2$.
3. There exists an $n$ x $n$ QH matrix for all $n$ except for $n$ =1 mod 8.
4. $R_Q(n-1)=n$ ? whenever an $n$ x $n$ Hadamard matrix exists.
5. $R_Q(n)$ = ? in general
6. The number of inequivalent $m$ x $n$ QH classes for $2<m\leq R_Q(n)$ ?
7. $N_E(m,n)$ = ?
8. The number of $n$ x $n$ H-minimals or symmetric H-minimals?

# Any question ?

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