Locally Repairable Fractional Repetition Codes

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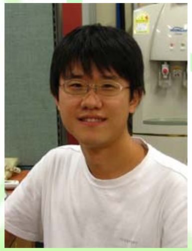
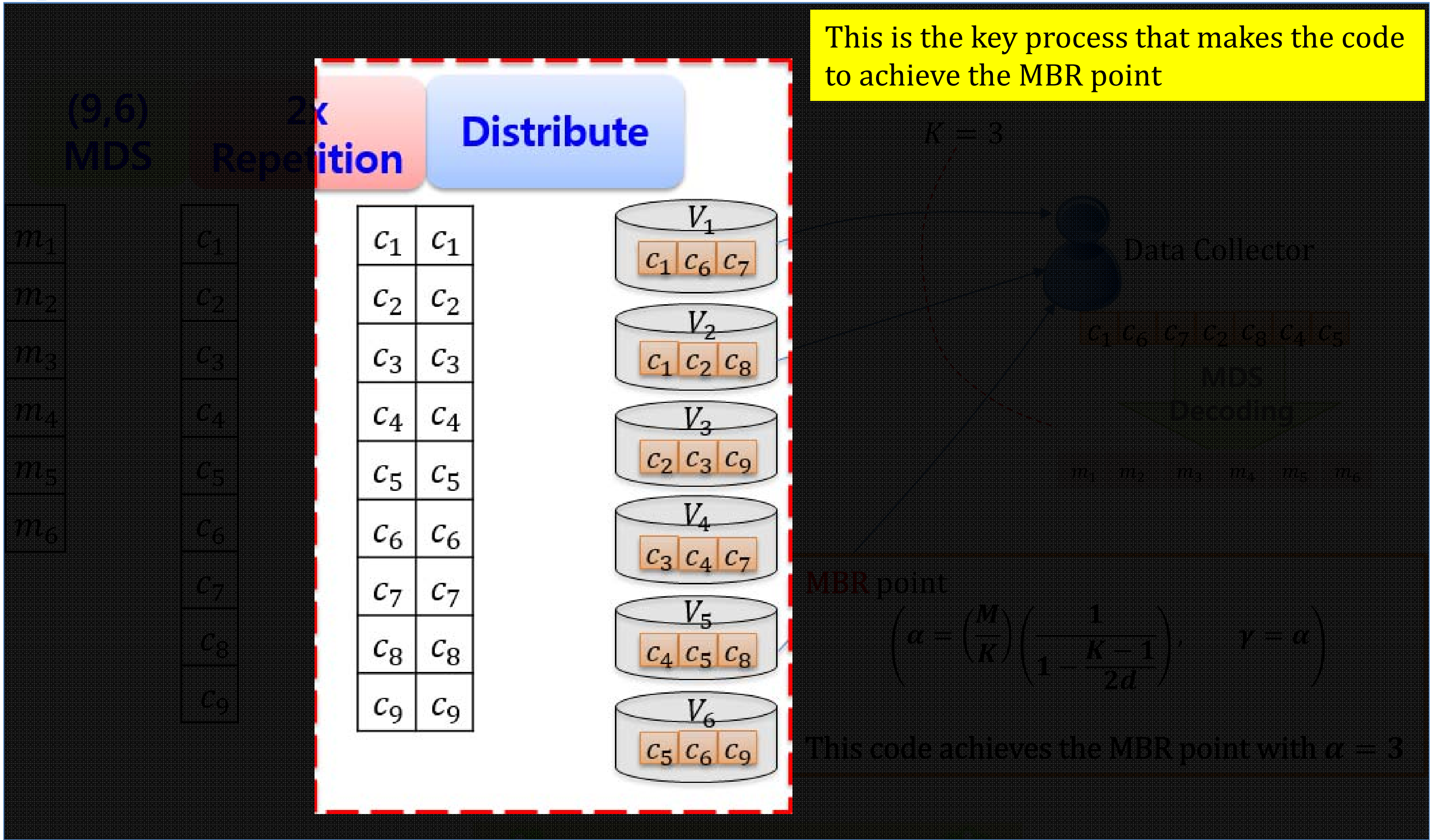



2015 International Workshop on Signal Design and Its Applications (IWSDA 2015)


2007 AAECC
at the campus of IISc,
8 year ago
Congratulation
Is he ?
Am I going to
be 60 years old
very soon?
(8 years later??)
yes, of course....

Fractional Repetition Codes

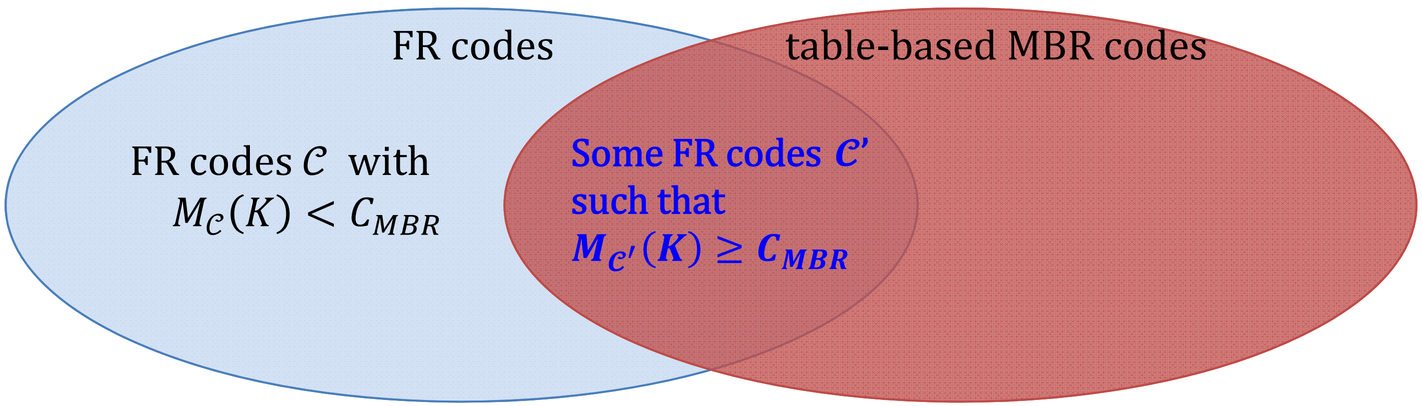
[S. E. Rouayheb-2010]



The maximum file size of FR Codes

- $M_{\mathcal{C}}(K)$ of FR codes $\mathcal{C}$
  - The maximum file size that can be stored by the FR code given K
  - For FR codes, this is the same as the maximum number of distinct symbols (or packets) that can be obtained by contacting any K nodes

- A Fractional Repetition Code whose maximum file size achieves the MBR capacity can be regarded as an MBR code
  - $M_{\mathcal{C}'}(K) \geq C_{MBR}$ for some FR codes $\mathcal{C}'$
    - $M_{\mathcal{C}'}(K) > C_{MBR}$ is possible due to the <u>table-based repair</u>
    - Strictly, FR codes are not the same as the MBR codes (random repair)



FR Codes and MBR Codes

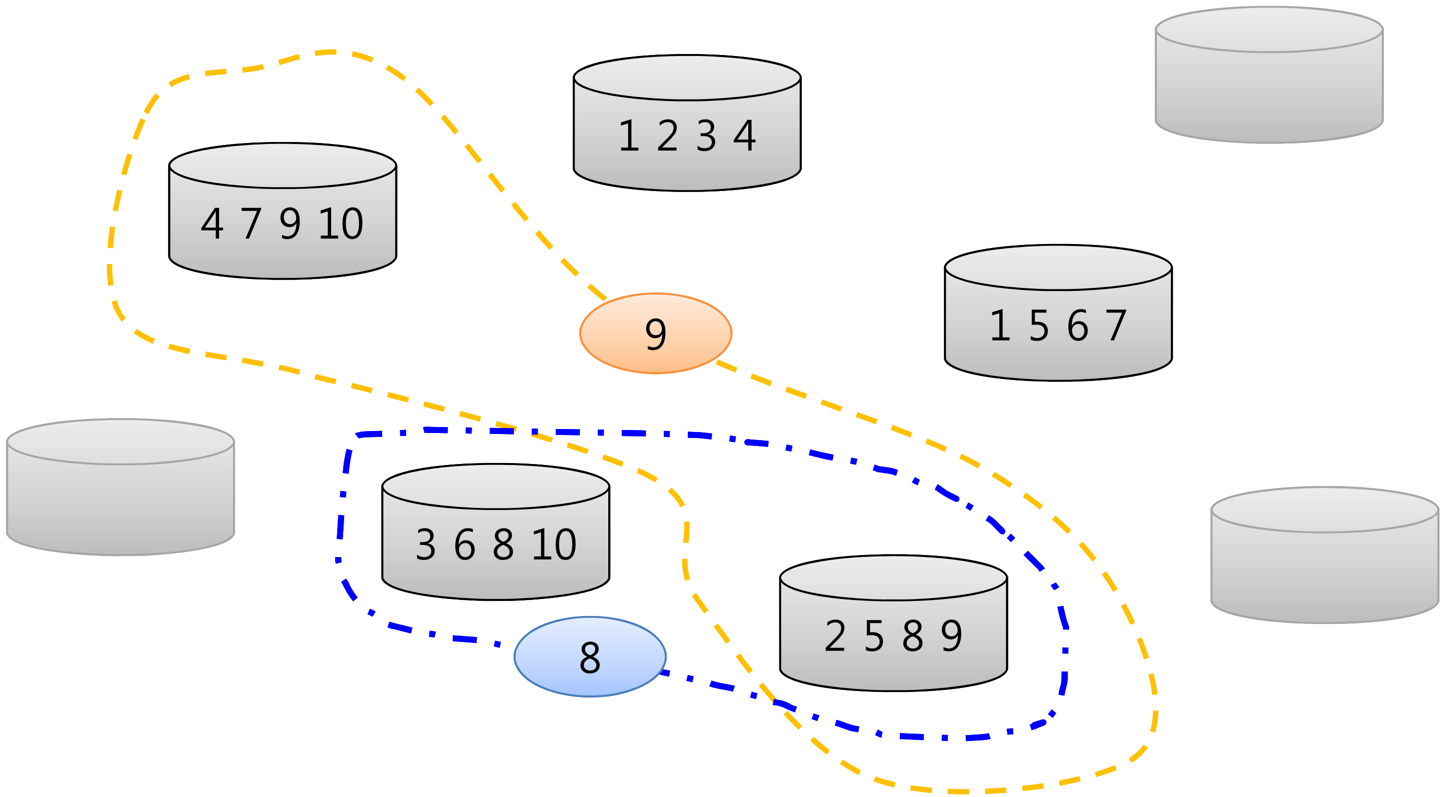
- There are many explicit constructions for FR codes $\mathcal{C}'$ with the maximum file size that satisfies the following:

$$M_{\mathcal{C}'}(K) \geq C_{MBR}$$

  - [S. E. Rouayheb-2010]: Graphs, Steiner systems
  - [J. C. Koo-2011]: Finite geometries, Bipartite cage graphs
  - [S. Pawar-2013]: Balls-in-bins for Randomized construction
  - [O. Olmez-2013] : Resolvable Designs, Mutually Orthogonal Latin Squares
  - [Z. Bing-2014]: Group Divisible Designs

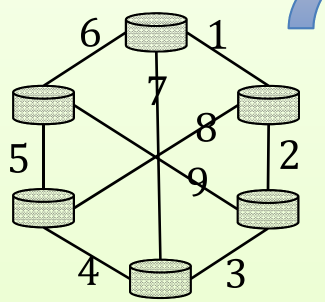
Constraints for achieving MBR capacity

- Traditional FR Codes
  - Every pair of nodes can store at most 1 symbol in common.
  - From this construction, the FR code can achieve the MBR capacity.
  - This is an explicit construction for MBR codes

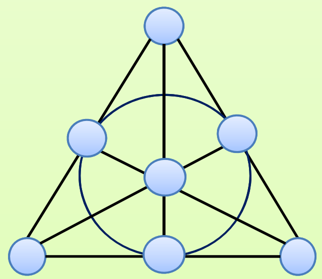


Locality of FR Codes

This construction guarantees that the resulting code to be an MBR code



OR

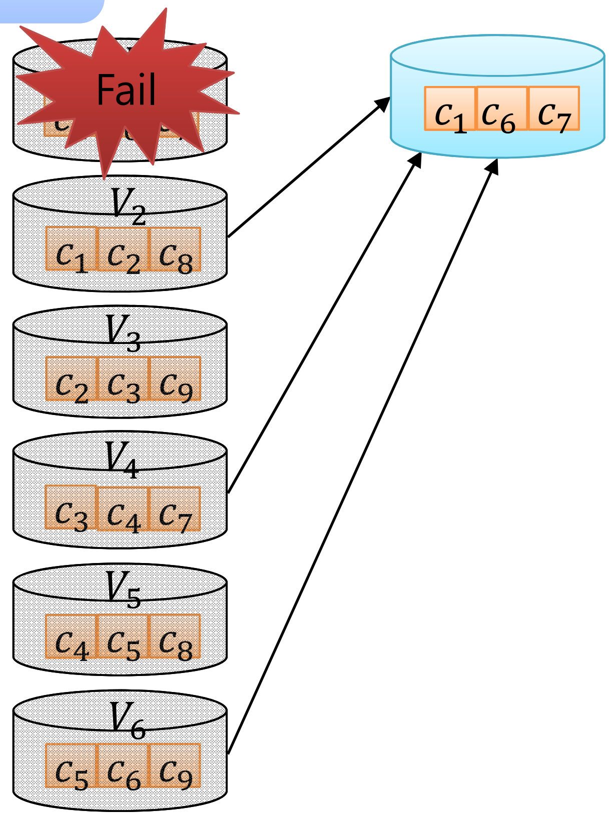


⋮

Block designs

Distribute

c_1	c_1
c_2	c_2
c_3	c_3
c_4	c_4
c_5	c_5
c_6	c_6
c_7	c_7
c_8	c_8
c_9	c_9

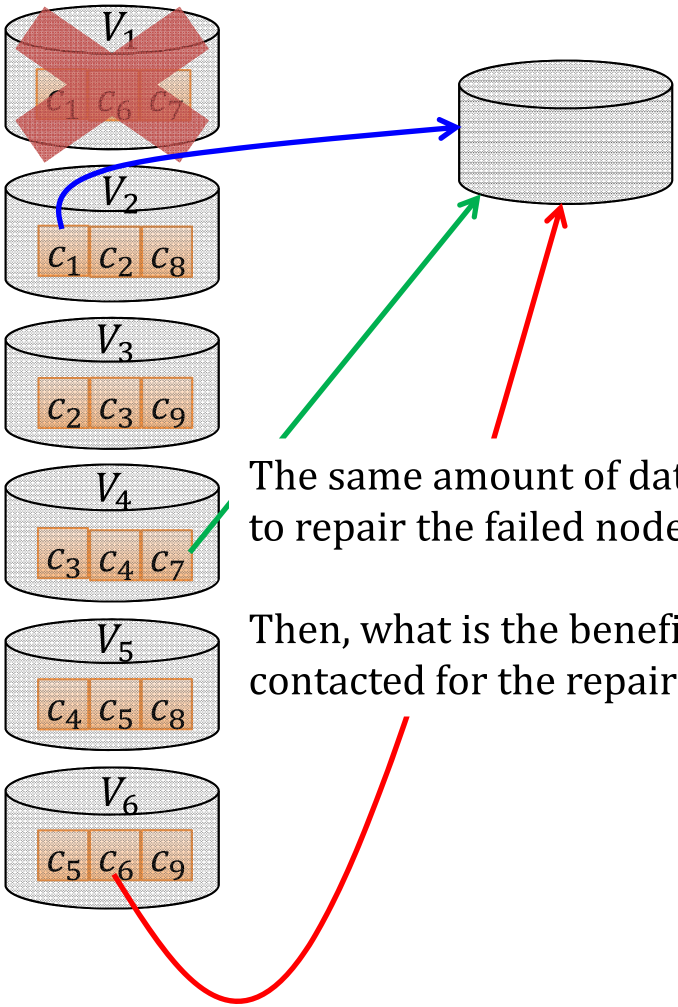


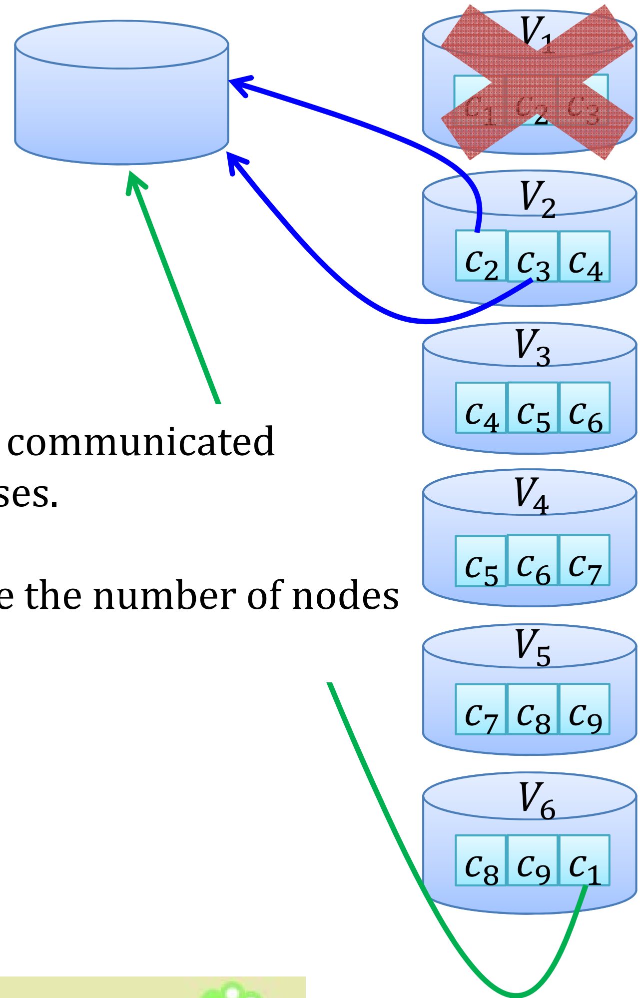
Locality of the FR codes
from this construction is always α

Locality of FR Codes

Original FR Codes

Proposed
Locally Repairable FR Codes





The same amount of data should be communicated to repair the failed node for both cases.

Then, what is the benefit to decrease the number of nodes contacted for the repair?

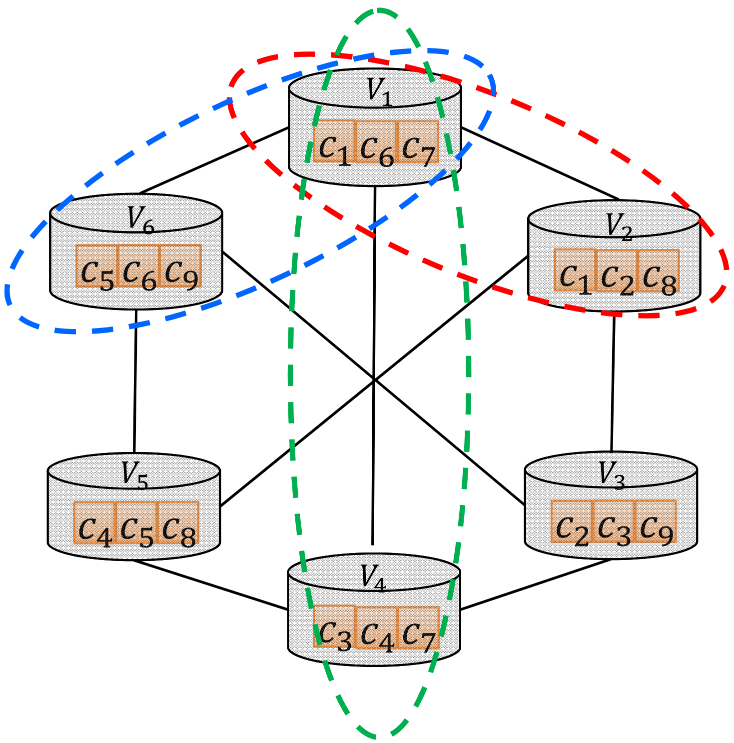
Multi-Node Failure

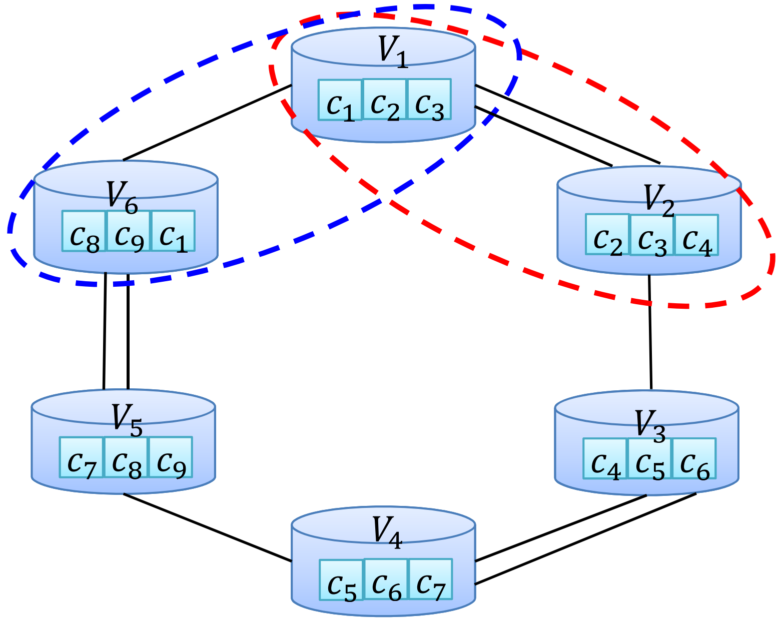
- Locally unrepairable 2-node failure patterns

Original FR Codes

Reduce locality

Proposed
Locally Repairable FR Codes





3 locally unrepairable 2-node failure patterns including node V_1

2 locally unrepairable 2-node failure patterns including node V_1

Locally Repairable FR Codes

- **Definition 1.** **[S. E. Rouayheb-2010]**

  A *Fractional Repetition (FR) code* $\mathcal{C}$ with repetition degree ρ, for an (n, K, d) DSS, is a collection $\mathcal{C}$ of n subsets $V_0, \ldots, V_{n-1}$ of a set $\Omega = \{0, \ldots, \theta - 1\}$ and of cardinality d each, satisfying the condition that each element of Ω belongs to exactly ρ sets in the collection.

  Note that $d = \alpha$ in this case

- **Definition 2.** An (n, K, d, α) *locally repairable FR code* is the (n, K, d, α) FR code with the repair degree d which is smaller than the storage size α.

Locally Repairable FR Codes

- **Theorem 1.** The maximum file size $M_{LFR}(K)$ of an $(n, K, d = 2, \alpha > 2)$

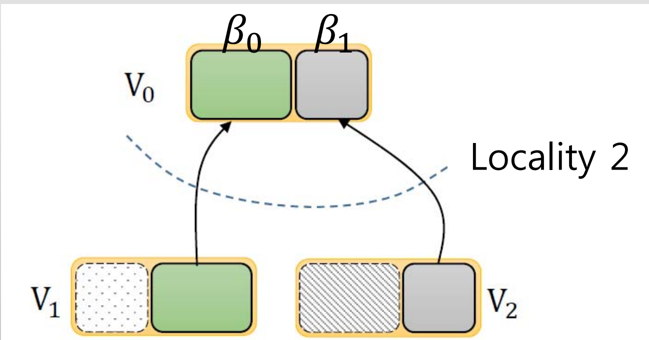
  Locally repairable FR code satisfies:

This is the maximum number of distinct symbols that can be obtained by contacting any ***K*** nodes

$$M_{LFR}(K) \leq \alpha + \underbrace{\{(\alpha - \beta_0) + (\alpha - \beta_1) + (\alpha - \beta_0) + \cdots\}}_{(K-1)-\text{terms}}$$

where $\alpha = \beta_0 + \beta_1$ and $\beta_0 \geq \beta_1$.

proof



K	$M(K)$
1	3
2	3 + 1
3	3 + 1 + 2
4	3 + 1 + 2 + 1
5	3 + 1 + 2 + 1 + 2

Proposed Constructions

- **Construction 1** (attains the bound of Theorem 1)
  - Repetition degree $\rho = 2$

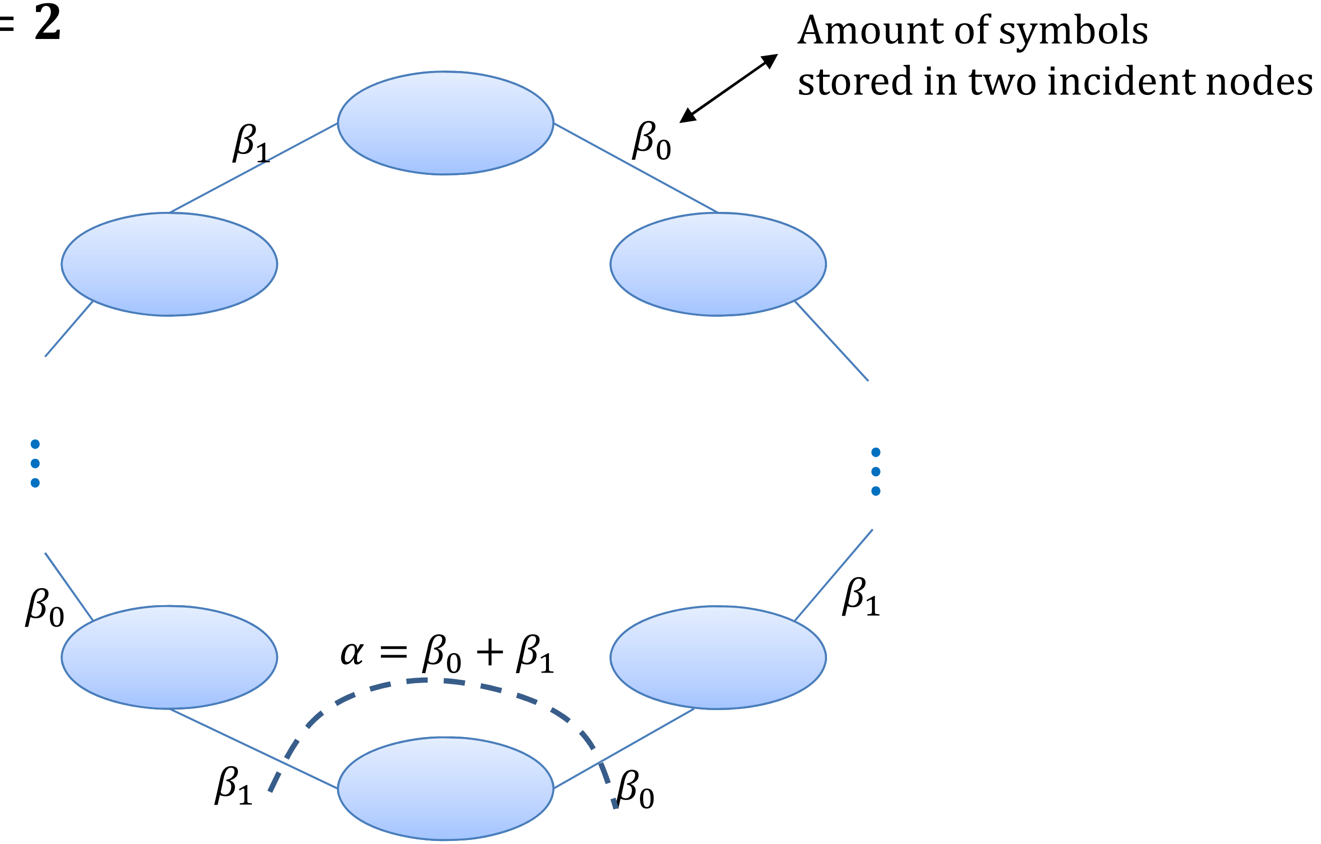
- **Construction 2** (attains the bound of Theorem 1)
  - Repetition degree $\rho = 3$
  - Large number of storage nodes are required

- **Construction 3**
  - Repetition degree $\rho = 3$
  - Reduces the number of storage nodes
  - But does not attain the capacity bound

Construction 1

Goal:
1. locality $d = 2$
2. achieves the capacity

- **Only for $\rho = 2$**



➢ $M_{con1}(K, \alpha) = \alpha + (\alpha - \beta_0) + (\alpha - \beta_1) + (\alpha - \beta_0) + \cdots$

if there is no cycle of size $\leq K$

$\Leftrightarrow n \geq K + 1$

Some Possible Parameters for Construction 1

$\rho\theta = n\alpha$: Condition for all FR codes

$2\theta = n\alpha$ — Only $\rho = 2$ is possible for construction 1 for each $\alpha = 3,4,5,\ldots$

Varying n, θ for fixed α

α	θ	$n = \frac{2\theta}{\alpha}$	$K \leq n-1$	$M(K) = \alpha + \underbrace{\{\beta_1 + \beta_0 + \beta_1 + \cdots\}}_{(K-1)-times}$	MDS code parameter $(\theta, M(K))$
3	3	$\frac{2\theta}{3} = 2$	1	3	(3,3)
3	6	4	1	3	(6,3)
			2	$3+1$	(6,4)
			3	$3+1+2$	(6,6)
3	9	6	1	3	(9,3)
			2	$3+1$	(9,4)
			3	$3+1+2$	(9,6)
			4	$3+1+2+1$	(9,7)
			5	$3+1+2+1+2$	(9,9)
3	12	8	⋮	⋮	⋮

Some Possible Parameters for Construction 1

$\rho\theta = n\alpha$ → $2\theta = n\alpha$

Only $\rho = 2$ is possible for construction 1

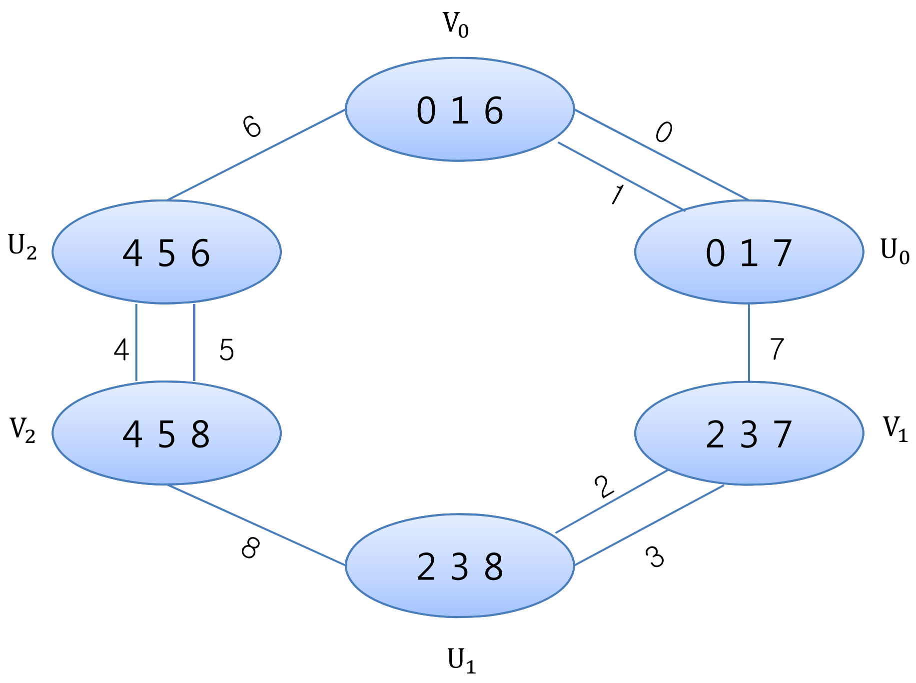
Varying α

$\boldsymbol{\alpha}$	$\boldsymbol{\theta}$	$\boldsymbol{n = \frac{2\theta}{\alpha}}$	$\boldsymbol{K \leq n-1}$	$\boldsymbol{M(K) = \alpha + \underbrace{\{\beta_1 + \beta_0 + \beta_1 + \cdots\}}_{(K-1)-times}}$	MDS code parameter $\boldsymbol{(\theta, M(K))}$
4	12	$\frac{2\theta}{4} = 6$	1	4	$(12, 4)$
			2	$4+2$	$(12, 6)$
			3	$4+2+2$	$(12, 8)$
			4	$4+2+2+2$	$(12, 10)$
			5	$4+2+2+2+2$	$(12, 12)$
5	15	$\frac{2\theta}{5} = 6$	1	5	$(15, 5)$
			2	7	$(15, 7)$
			3	10	$(15, 10)$
			4	12	$(15, 12)$
			5	15	$(15, 15)$
⋮	⋮	⋮	⋮	⋮	⋮

Construction 1

Goal:
1. locality $d = 2$
2. achieves the capacity

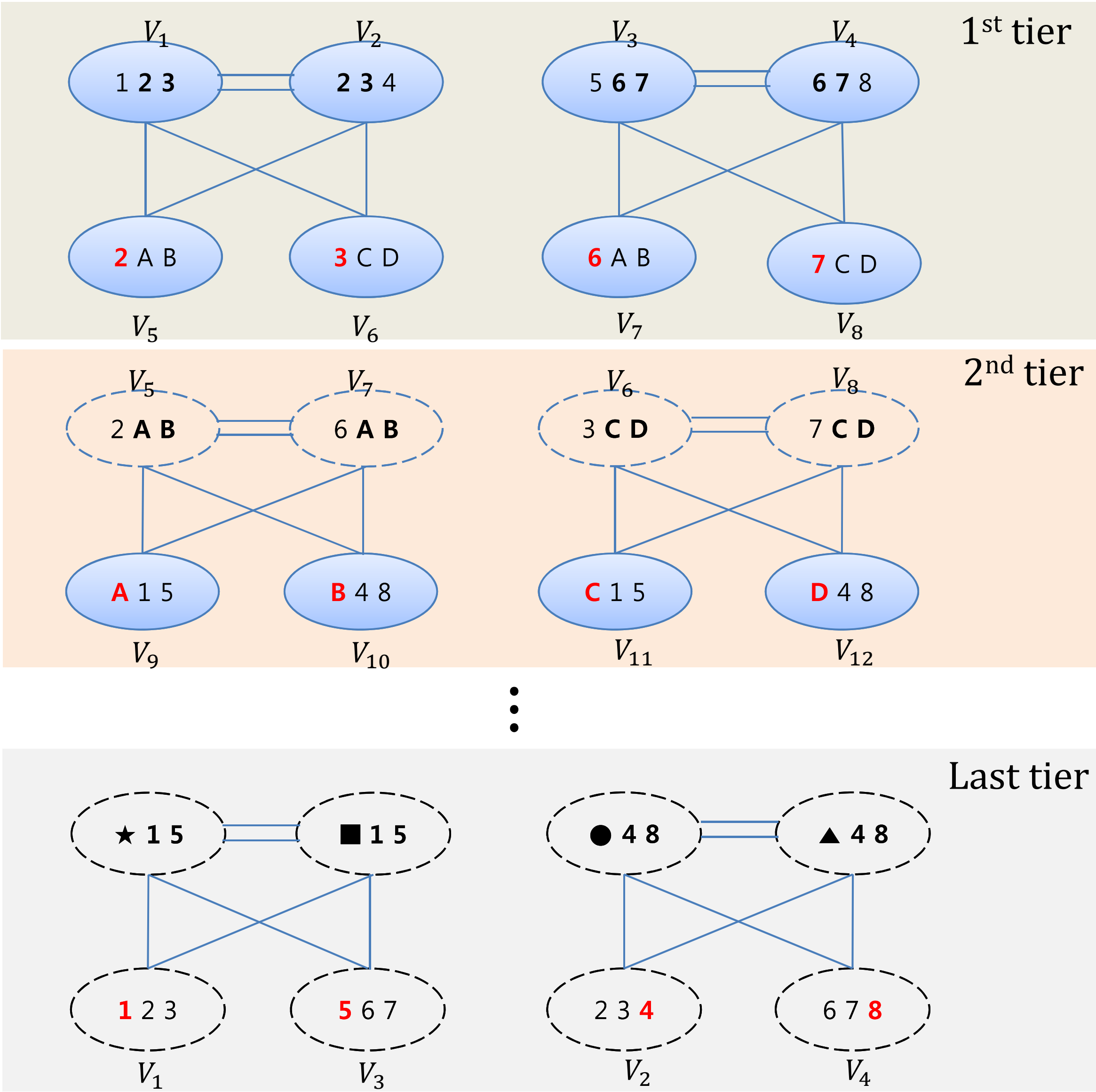
- **Example**: $\alpha = 3, \theta = 9, n = 6$
  - $M_{con1}(K) = 3 + (1 + 2 + 1 + \cdots \quad)$ ← K-1 terms here
    only for the value of $K = 1,2, \ldots, 5 < n$



Construction 2

- **Only for $\rho = 3$ and $\alpha = 3$**

Goal:
1. locality $d = 2$ for single failure
2. local repair for double-failure
3. achieves the capacity



Construction 2

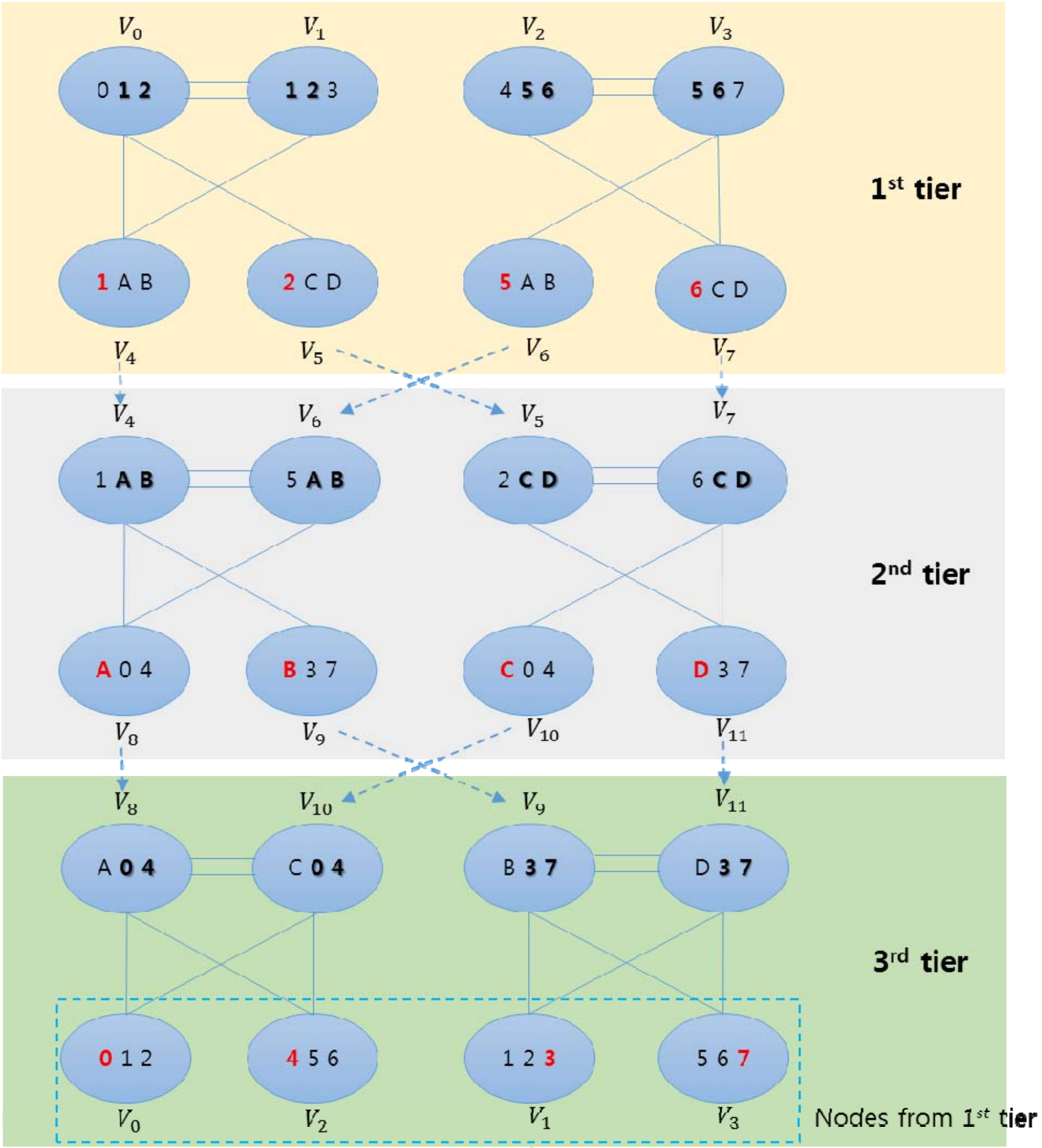
- Construction for $K \leq 4$
- If $K \geq 5$,
  then the code of the figure cannot achieve the capacity $M_{LFR}(K)$ of Theorem 1
- To achieve the capacity $M_{LFR}(K)$, the number of tiers l should satisfy that

$$l = \max\left\{3, \left(\left\lceil \frac{K}{2} \right\rceil + 1\right)\right\}$$

- Then the number of nodes

$$n = 4l$$

Goal:
1. locality $d = 2$ for single failure
2. local repair for double-failure
3. achieves the capacity

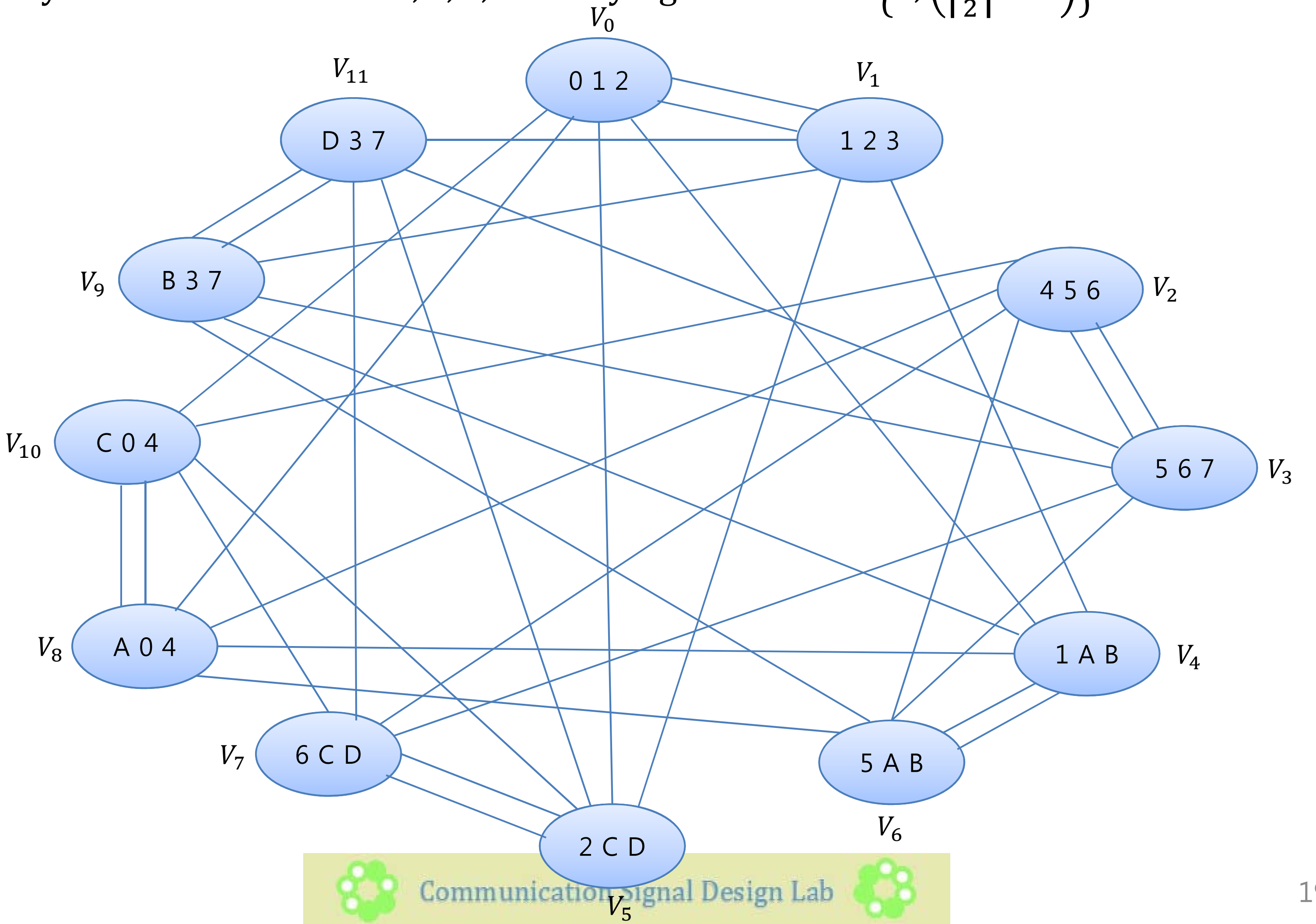


Construction 2

Goal:
1. locality $d = 2$ for single failure
2. local repair for double-failure
3. achieves the capacity

- **Example** : $\theta = n = 12$
  - $M_{con2}(K) = 3 + (1 + 2 + 1 + \cdots \quad)$ ← K-1 terms here

    only for the value of $K = 1, 2, 3, 4$ satisfying $n = 4 \cdot \max\left\{3, \left(\left\lceil\frac{K}{2}\right\rceil + 1\right)\right\}$



Some Possible Parameters for Construction 2

$\rho\theta = n\alpha$: Condition for all FR codes

Only $\rho = 3$ and $\alpha = 3$ is possible for construction 2

$3\theta = 3n$

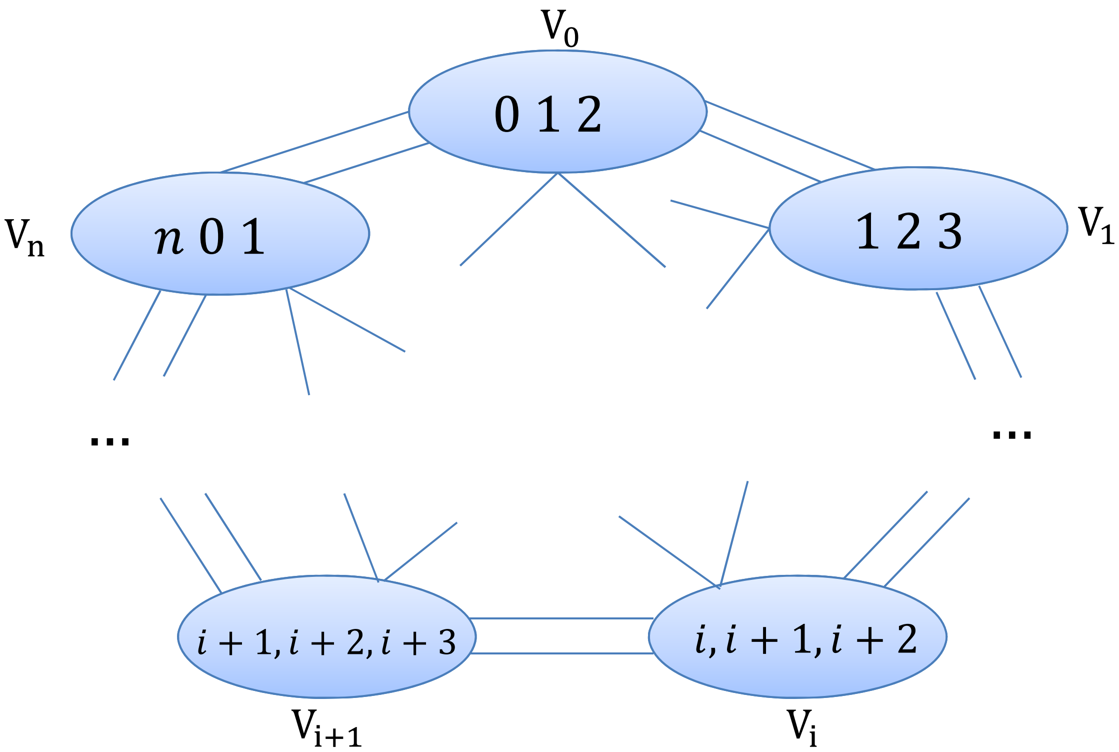
Varying n, θ

α	θ	$n = \theta$ multiple of 4	K	$M(K) = \alpha + \underbrace{\{\beta_1 + \beta_0 + \beta_1 + \cdots\}}_{(K-1)-times}$	MDS code parameter $(\theta, M(K))$
3	12	12	1	3	(12, 3)
			2	3 + 1	(12, 4)
			3	3 + 1 + 2	(12, 6)
			4	3 + 1 + 2 + 1	(12, 7)
3	16	16	1	3	(16, 3)
			2	3 + 1	(16, 4)
			3	3 + 1 + 2	(16, 6)
			4	3 + 1 + 2 + 1	(16, 7)
			5	3 + 1 + 2 + 1 + 2	(16, 9)
			6	3 + 1 + 2 + 1 + 2 + 1	(16, 10)
⋮	⋮	⋮	⋮	⋮	⋮

Construction 3

Goal:
1. locality $d = 2$ for single failure
2. local repair for double-failure
3. reduce n

- **Only for $\rho = 3$ and $\alpha = 3$**
  - allows smaller number of nodes than construction 2.
  - This cannot achieve the capacity of Theorem 1.



Some Possible Parameters for Construction 3

$\rho\theta = n\alpha$: Condition for all FR codes

Only $\rho = 3$ *and* $\alpha = 3$ is possible for construction 3

$3\theta = 3n$

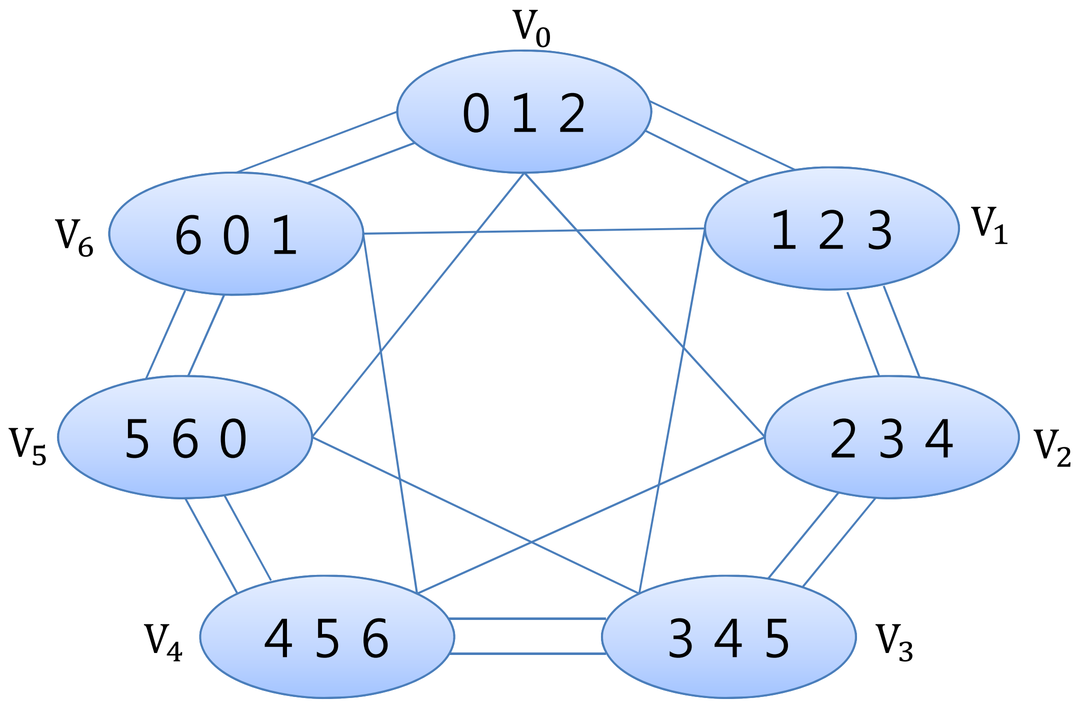
Varying n, θ for fixed α

α	θ	$5 \leq n = \theta$	$K \leq n-2$	$M(K) = \alpha + (K-1)$	MDS code parameter $(\theta, M(K))$
3	5	5	1	3	$(5,3)$
			2	$3+1$	$(5,4)$
			3	$3+1+1$	$(5,5)$
3	6	6	1	3	$(6,3)$
			2	$3+1$	$(6,4)$
			3	$3+1+1$	$(6,5)$
			4	$3+1+1+1$	$(6,6)$
3	7	7	1 ⋮ 5	$3+(\underbrace{1+1+\cdots}_{(K-1)-times})$	$(7, M(K))$
⋮	⋮	⋮	⋮	⋮	⋮

Construction 3

Goal:
1. locality $d = 2$ for single failure
2. local repair for double-failure
3. reduce n

- **Example**: $\theta = 7, n = 7$
  - $M_{con3}(K) = 3 + 1 + 1 + 1 + \cdots$
    $= 3 + (K - 1) < M_{LFR}(K)$



Comparison with (other) FR codes

- $\alpha = 3$

$\rho = 2$

	FR codes [2010] from regular graph	Construction 1
max. file size $M(K)$	$K\alpha - \binom{K}{2}$	$K\alpha - (K-1) - \left\lceil \frac{K-1}{2} \right\rceil$
# nodes (n)	n	$n = \max\{K+1, 4\}$, n is even
Locality (1-failure)	3	**2**
Locality (2-failure)	MDS dec.	MDS dec.

$\rho = 3$

	FR codes [2010] from Steiner system	Construction 2	Construction 3
max. file size $M(K)$	$K\alpha - \binom{K}{2}$	$K\alpha - (K-1) - \left\lceil \frac{K-1}{2} \right\rceil$	$\alpha + (K-1)$
# node (n)	n	$n = 4 \cdot \max\left\{3, \left\lceil \frac{K}{2} \right\rceil + 1\right\}$	$n = K + 2$
Locality (1-failure)	3	**2**	**2**
Locality (2-failure)	3	**3**	**2**

Comparison ($\alpha = 3$) with other LRC

	2x Repetition Code	Construction 1	Simple LRC [Papailiopoulos 2014]
Locality (1-failure)	1	2	2
Repair bandwidth	3	**3**	6
Unrecoverability	3.33×10^{-4}	$\mathbf{5.94 \times 10^{-8}}$	1.47×10^{-7}
Minimum distance	2	4	4
MTTDL Mean Time To Data Loss	66.25 days	**32.69 years**	7.17 years
# Computations per single repair	NONE	**NONE**	3 adds/node
Storage overhead	$1\times$	$2\times$	$2\times$

For the MTTDL calculation, we used a standard Markov model.

Simple LRC [D. S. Papailiopoulos-2014]

- Simple but more reliable than the repetition
- Some additions are required for node repair

Communication Signal Design Lab

Comparison Summary

- New LRCs are proposed based on Fractional Repetition codes

Compared to the original FR Codes	• **Better locality**
	• Less capacity • Restricted α
Compared to the other LRCs	• **Computations are NOT required for a node repair** • **Minimum repair bandwidth** • **Larger MTTDL can be achieved (More reliable)**
	• Not d_{min}-optimal

- Construction 1 gives a code that attains the bound of Theorem 1
  - for $\rho = 2$ only, but with $\alpha = 3,4,5, \ldots$

- Construction 2 gives a code that attains the bound of Theorem 1
  - for $\rho = 3$ and $\alpha = 3$ only

Selected Open Problems

What happens if we allow $\rho \geq 3$ and/or $\alpha > 3$?

- Is there any construction that attains the bound of Theorem 1?
- We do not have any known constructions in this case.
- If one can prove that none exists, then it implies
  - the bound of Theorem 1 is not tight, and
  - A new bound should be derived