Locally Repairable Codes with Locality 1 and Availability

CSDL

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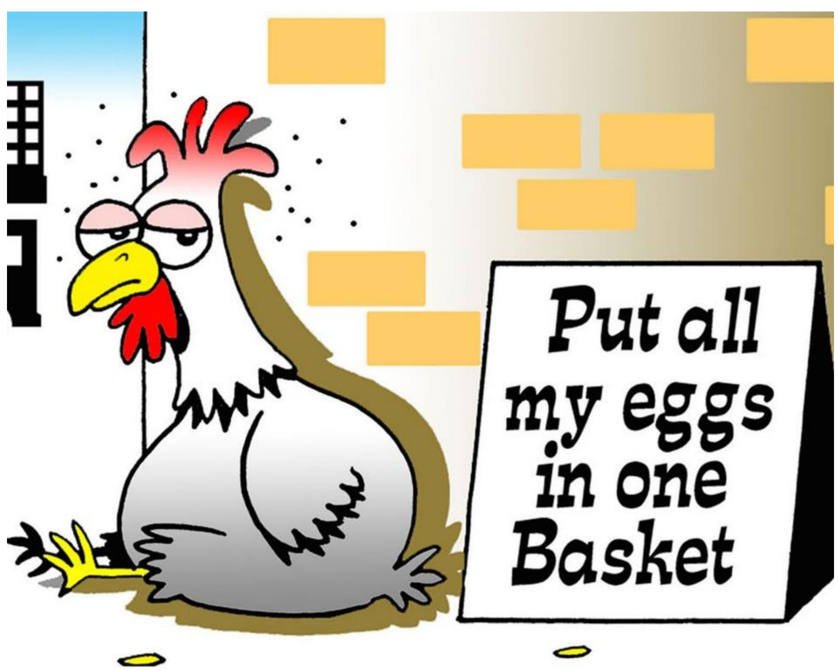
Data Storage

- Store a file DATA in a storage device

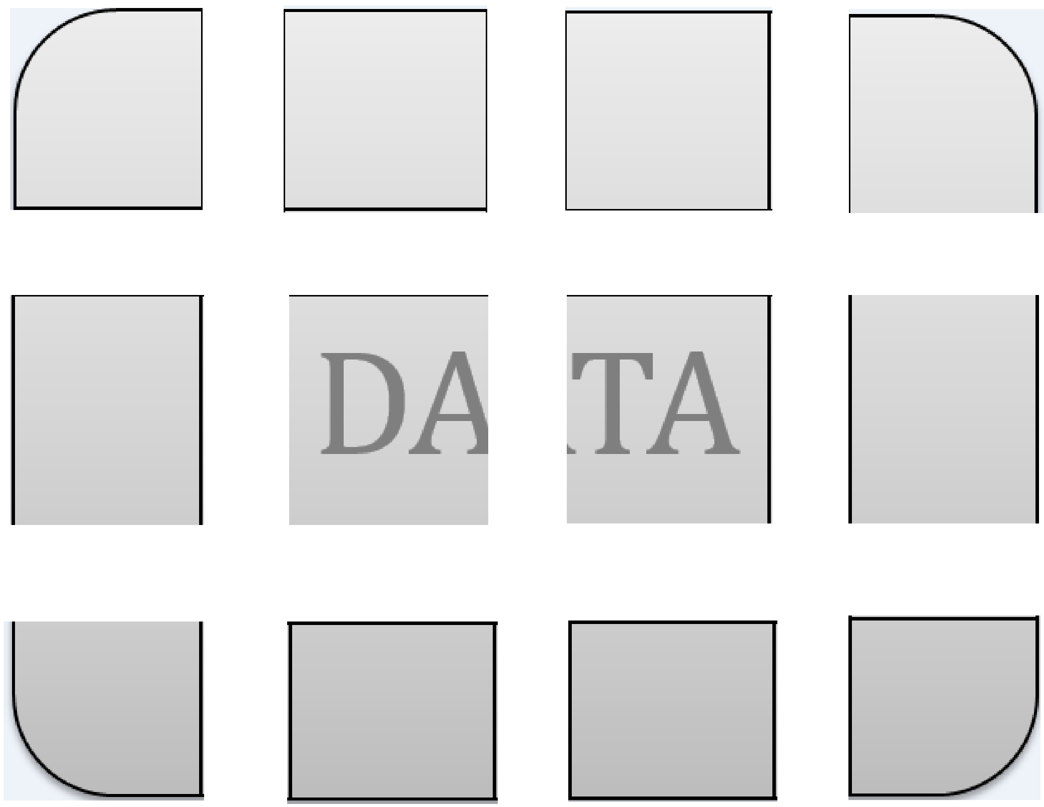


Data Storage

- Encode the file DATA
  - Add redundancy

Data Distribution

- Distribute the encoded file to n storage nodes



Data Distribution

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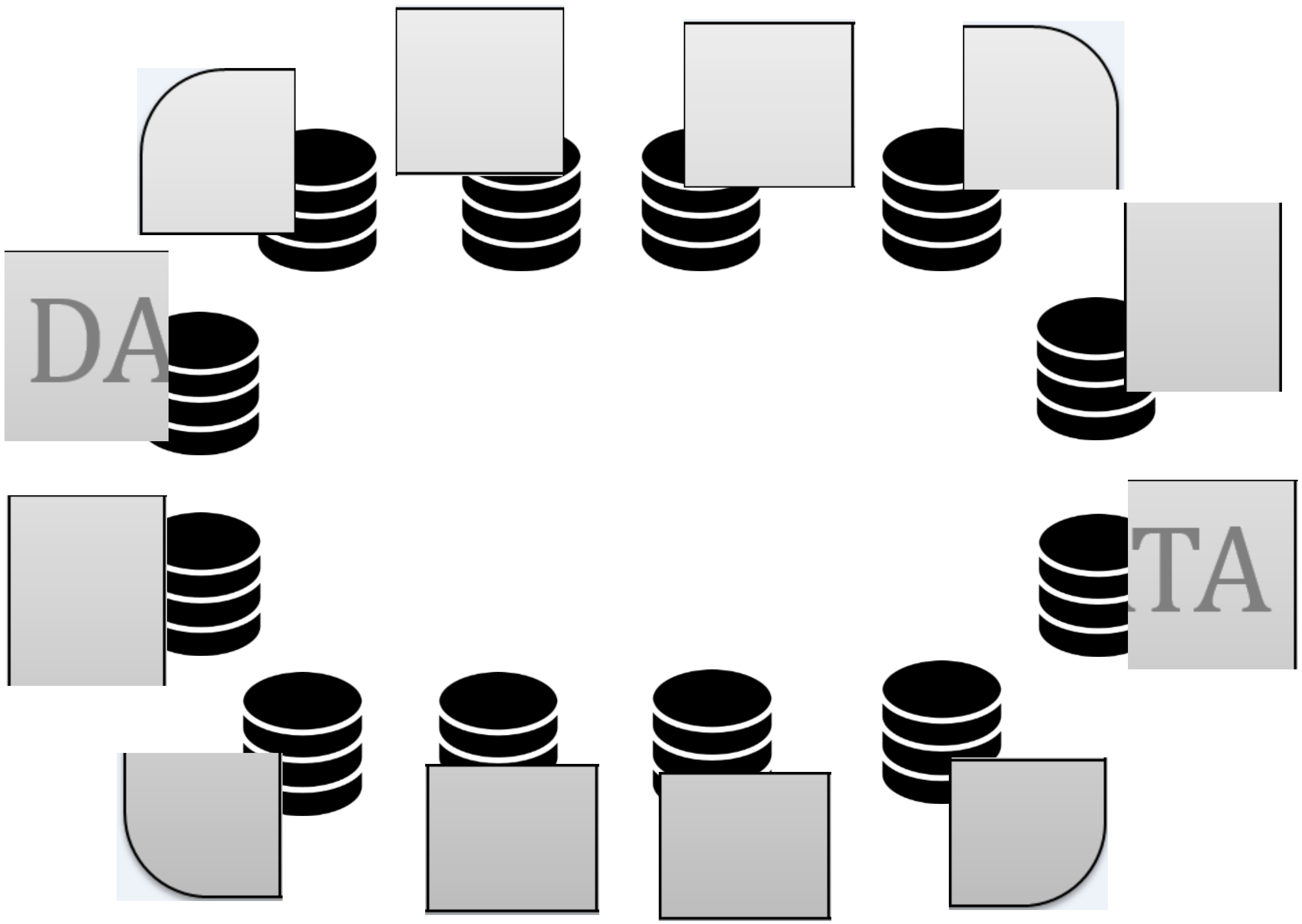
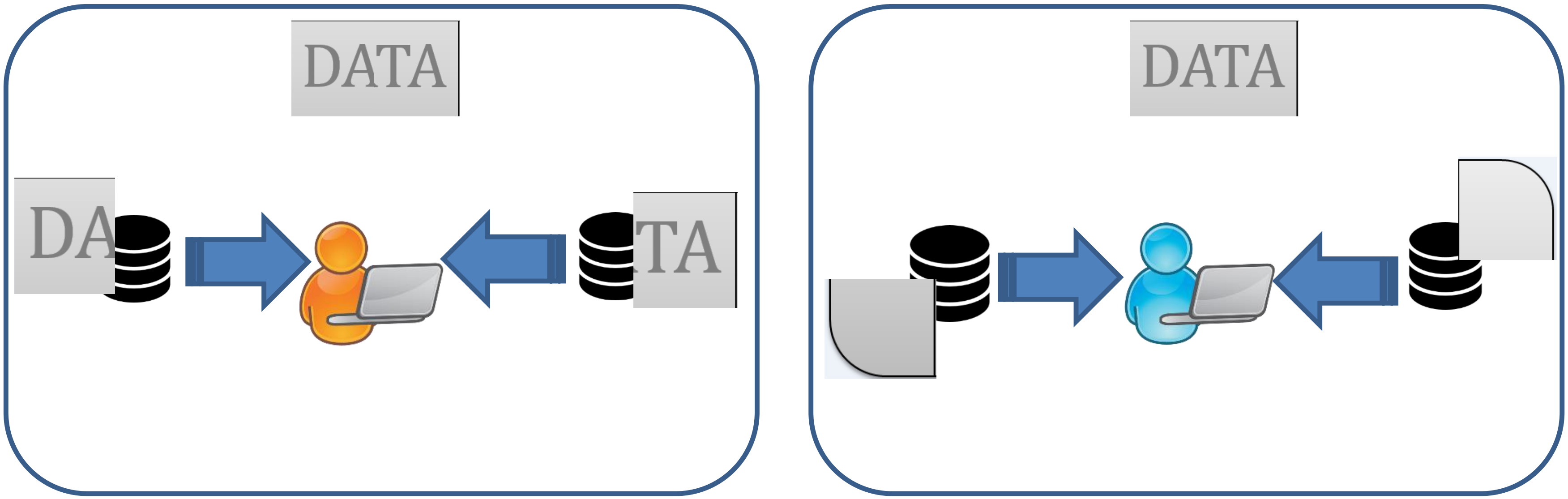


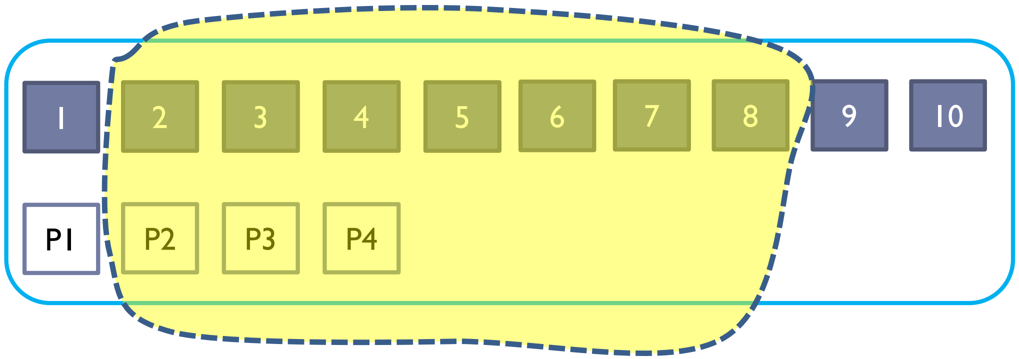
Data Collector

- Data Collector can retrieve the original file DATA by downloading from **any** ***K*** storage nodes



MDS Code

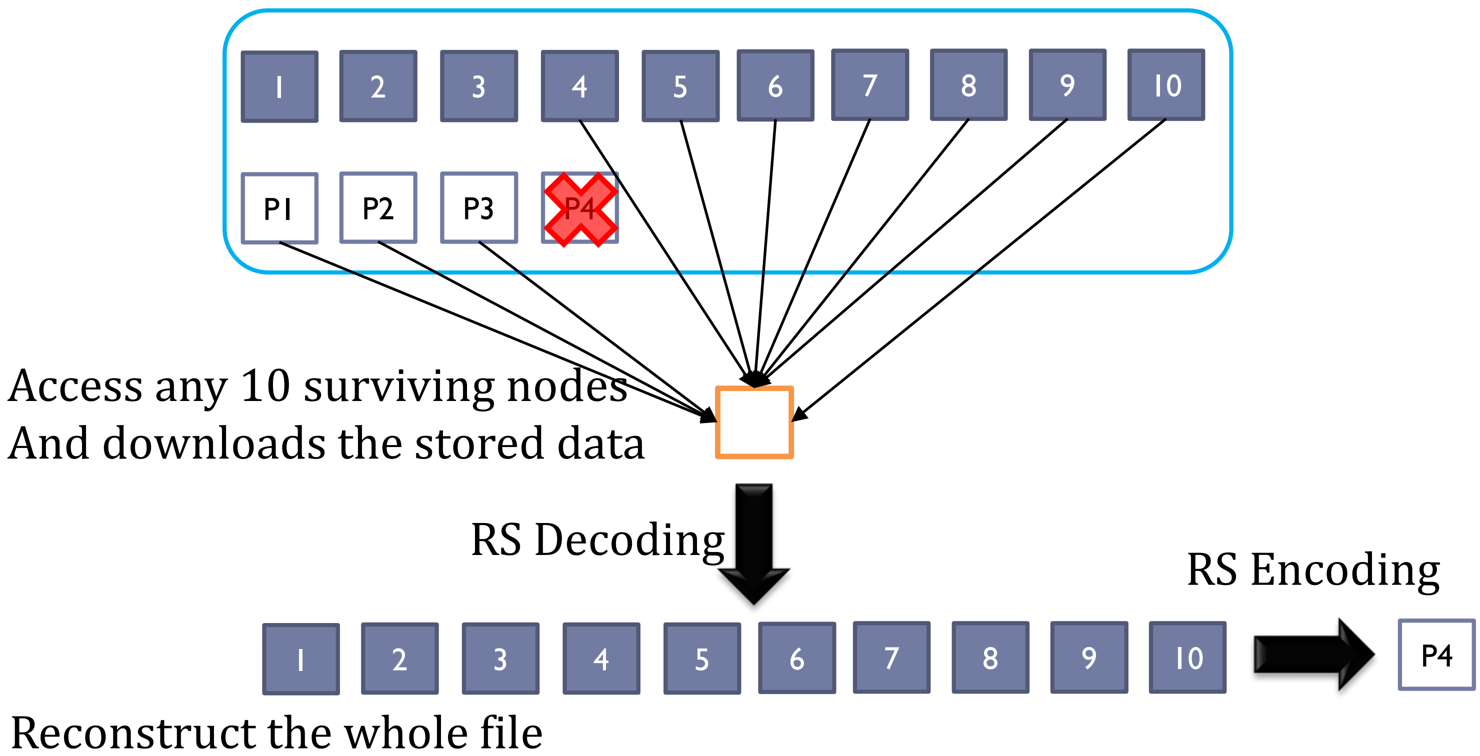
- Reed-Solomon (RS) Codes
  - Facebook introduced a (14, 10) RS code
  - This tolerate up to 4 missing blocks



The whole file can be reconstructed from any 10 coded blocks

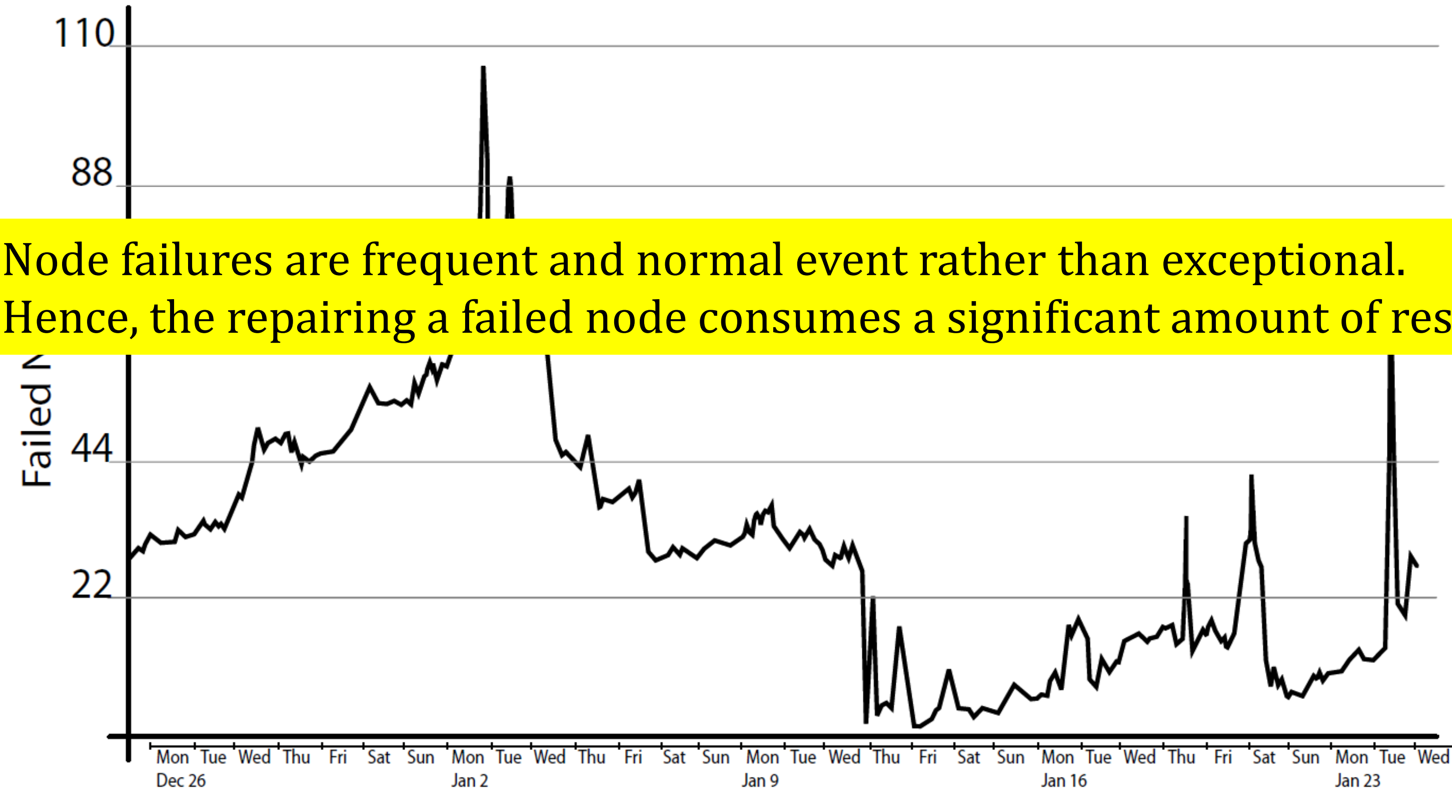
The Repair Problem

- Traditional erasure-correcting codes are optimized for recreation of the original message
  - But not for regeneration of individual lost encoded parts
  - Example: (14, 10) RS code



A Trace of Node Failures

The number of failed nodes over a single month in a 3000 node production cluster of Facebook



Node failures are frequent and normal event rather than exceptional.
Hence, the repairing a failed node consumes a significant amount of resources.

- More than 20 nodes fail daily on average
- Each node stores 15TB

Repair Metrics of Interest

- Repair bandwidth
  - The number of bits communicated in the network during a single failed node repair
- **Locality**
  - The number of nodes accessed to repair a single node failure

Locally Repairable Codes

- ✓ Let $\mathcal{C}$ be an $(n, k)_q$ code of length n, dimension k over a finite field $\mathbb{F}_q$
- ✓ The **locality** of the i-th coordinate of $\mathcal{C}$ is r if the value of the i-th symbol of a codeword of $\mathcal{C}$ is a function of r other coordinates and no such a set of coordinates of cardinality less than r exists
  - The set of such r coordinates that can repair the i-th symbol is called a **repair set**
- ✓ The locality of the code $\mathcal{C}$ is r if the symbol locality of every symbol in a codeword of $\mathcal{C}$ is at most r
- ✓ An (n, k) code $\mathcal{C}$ with locality $r \ll k$ is defined as a **locally repairable code**

Minimum Distance

- ✓ The minimum distance d of a code $\mathcal{C}$ is the minimum number of difference between every pair of two codewords in $\mathcal{C}$
- ✓ Erasure-correcting code with minimum distance d can tolerate up to $d-1$ erasures
- ✓ Singleton showed a bound on the best possible minimum distance of an (n,k) code:

$$d \leq n-k+1$$

- ✓ (n,k) codes that achieve the Singleton-bound are Maximum Distance Separable (MDS) codes
- ✓ There exists a locality-distance tradeoff
  - Any (n,k) code with locality r can have distance at most

$$d \leq n-k-\left\lceil \frac{k}{r} \right\rceil+2 \qquad \ldots(1)$$

  - Any MDS code must have trivial locality $r=k$

Availability

- ✓ If every symbol has t disjoint repair sets of size at most r, then such an LRC is said to have locality r and availability t
- ✓ The upper bound of the minimum distance of an LRC with locality r and availability t is

$$d \leq n - \sum_{i=0}^{t} \left\lfloor \frac{k-1}{r^i} \right\rfloor \qquad \ldots (2)$$

- ✓ LRCs which has the minimum distance that achieves the upper bound with equality is said to be **optimal**

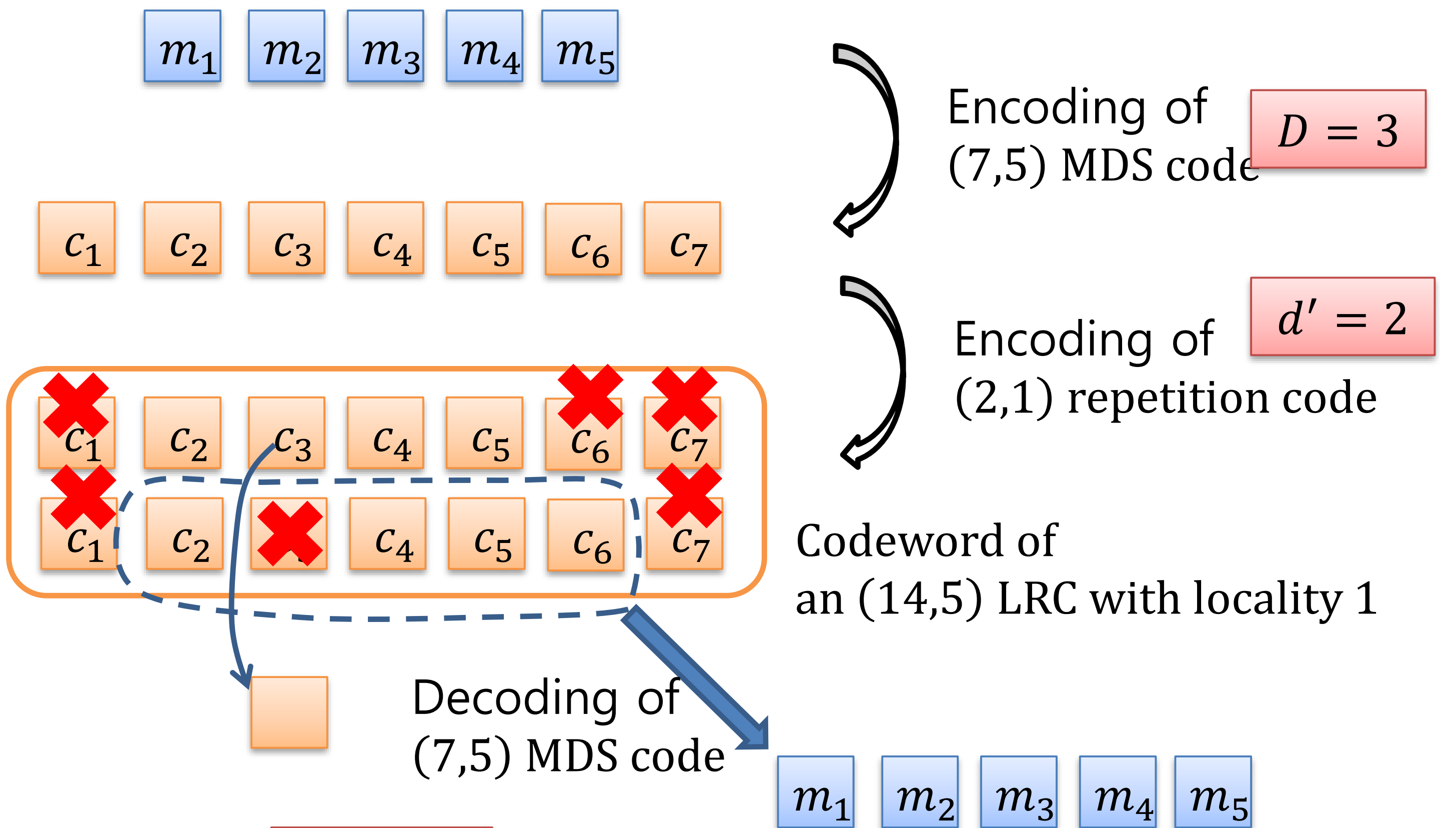
Construction for LRCs

- ✓ A unique solution for the locality 1 is the repetition code
  - ✓ A $(\rho, 1)$ repetition code replicates the symbol by $\rho - 1$ times
  - ✓ A $(\rho, 1)$ repetition code has minimum distance ρ and availability $\rho - 1$
  - ✓ Code rate of a $(\rho, 1)$ repetition code is $\frac{1}{\rho}$
- ✓ Concatenating with a code having large minimum distance makes the concatenated code to have larger minimum distance
- ✓ LRCs from serial concatenation
  - ✓ The inner code determines the locality
  - ✓ The minimum distance d of the serially concatenated code is $d \geq d'D$, where d' and D is the minimum distance of an inner code and an outer code, respectively

Construction for LRCs

✓ Serial Concatenation

- (θ, M) MDS code + $(\rho, 1)$ repetition code



Can tolerate up to 5 erasures ➔ $d = 6$

Optimality

- ✓ The concatenation results in a $(\rho\theta, M)$ LRC $\mathcal{C}$ with locality 1 and availability $\rho - 1$
- ✓ The minimum distance d of $\mathcal{C}$ is $d = \rho(\theta - M + 1)$
- ✓ When $\rho = 2$,
  - ✓ The code $\mathcal{C}$ achieves the Singleton-like bound, $d \le n - k - \left\lceil \frac{k}{r} \right\rceil + 2$

    Since $d = 2(\theta - M + 1) = 2\theta - M - \left\lceil \frac{M}{1} \right\rceil + 2$
- ✓ When $\rho \ge 3$,
  - ✓ The code $\mathcal{C}$ achieves the bound (2), $d \le n - \sum_{i=0}^{t} \left\lfloor \frac{k-1}{r^i} \right\rfloor$

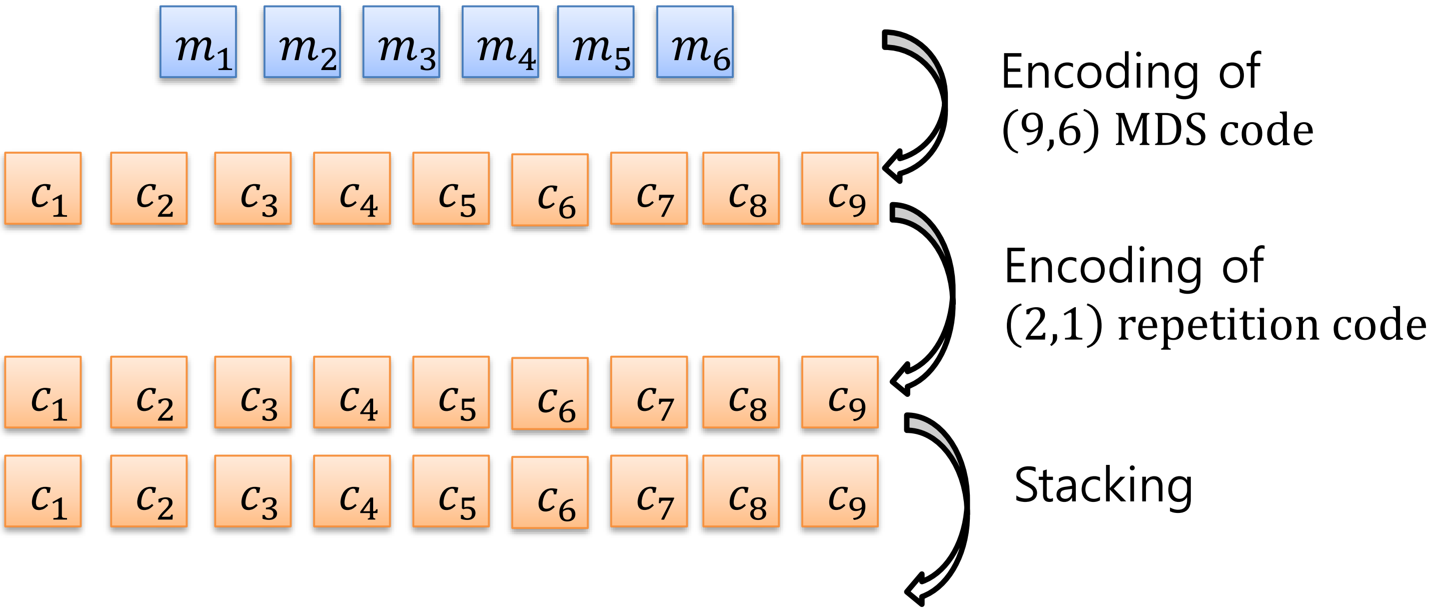
    since $d = \rho(\theta - M + 1) = \rho\theta - \sum_{i=0}^{\rho-1} \left\lfloor \frac{M-1}{1} \right\rfloor = \rho\theta - \rho(M-1)$

Comparisons

		Repetition code	Proposed (Concatenation)		RS code
$d = 4$	r	1	1		M
	t	3	2		0
	R	$\frac{M}{4M}$	$\frac{M}{2(M+1)}$		$\frac{M}{M+3}$
$d = 6$	r	1	1	1	M
	t	5	2	1	0
	R	$\frac{M}{6M}$	$\frac{M}{3(M+1)}$	$\frac{M}{2(M+2)}$	$\frac{M}{M+5}$

A Vector LRC from scalar LRCs

- ✓ A vector code is a code over a vector symbol alphabet $\mathbb{F}_q^{\alpha}$
- ✓ Let $\mathcal{C}$ be an $(n, M, \alpha, r)_q$ vector LRC
  - That takes a file of size M symbols in $\mathbb{F}_q$ encodes it to n blocks which contains α symbols of $\mathbb{F}_q$, and any erased block can be repaired by accessing at most r other blocks
- ✓ Simply stacking α scalar $(n, k, r)_q$ LRCs results in a vector $(n, M, \alpha, r)_q$ LRC



A Vector LRC from scalar LRCs

V_1	V_2	V_3	V_4	V_5	V_6
c_1	c_1	c_4	c_4	c_7	c_7
c_2	c_2	c_5	c_5	c_8	c_8
c_3	c_3	c_6	c_6	c_9	c_9

- ✓ An $(n, M, \alpha, 1)_q$ vector LRC $\mathcal{C}$ from α $(n, M, 1)$ scalar LRCs
  - ✓ has availability $t = \rho - 1$
  - ✓ the minimum distance d of $\mathcal{C}$ is $d = n - \left\lceil \frac{M}{\alpha} \right\rceil - \left\lceil \frac{tM}{r\alpha} \right\rceil + t + 1$
- ✓ The upper bound of the vector LRC is known only for $t = 1$
  - ✓ The bound is the same as the minimum distance of the proposed code
  - ✓ Therefore, the proposed vector LRC is optimal when $t = 1$

Conclusion

- We proposed an explicit construction for optimal LRCs
  - With locality 1 and arbitrary availability
  - Based on serial concatenation
- The study of constructions for LRCs with locality larger than 1 based on a concatenation will be an interesting future work
- Also, the construction of vector LRCs by stacking scalar LRCs in different ways will be a meaningful research topic