Alphabet-Dependent Upper Bounds for Locally Repairable Codes with Joint Locality

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Outline

- Motivation (joint locality)
- Alphabet-dependent bounds
- Singleton-like bounds
- Graph-based BLRCs
- Conclusions

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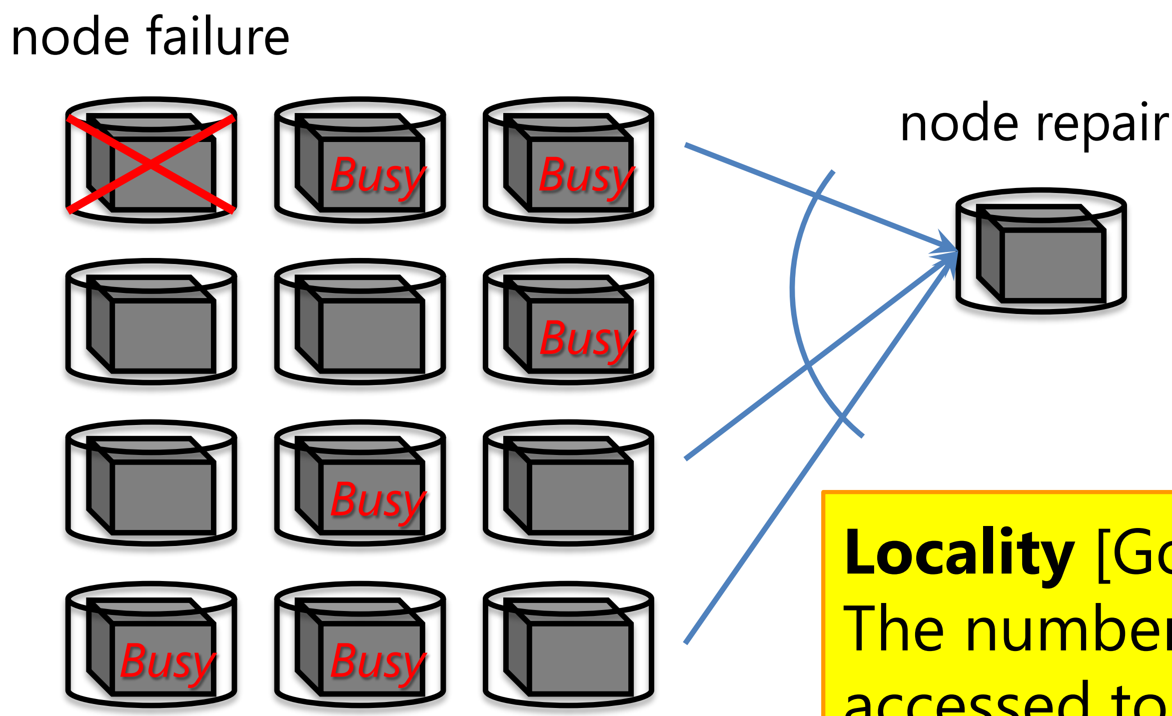
Motivation

- Locality in distributed storage systems

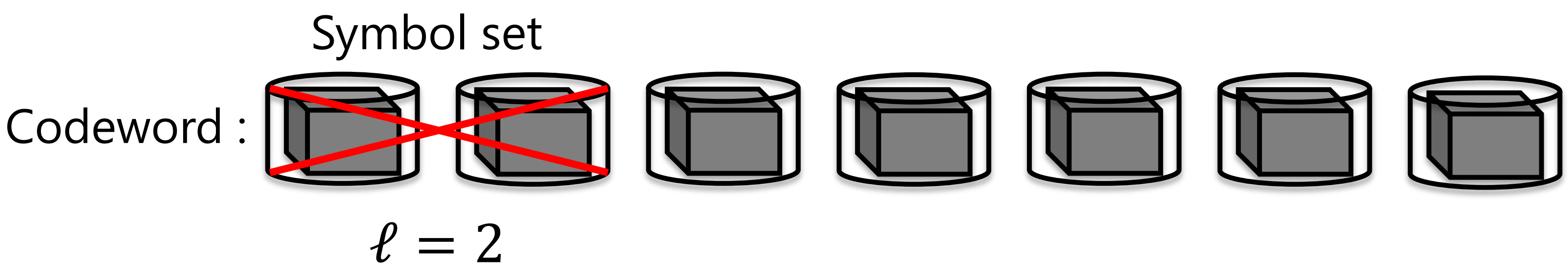


Locality [Gopalan et al. 12]
The number of nodes accessed to repair
one node failure

Motivation

- Generalized Definition of locality

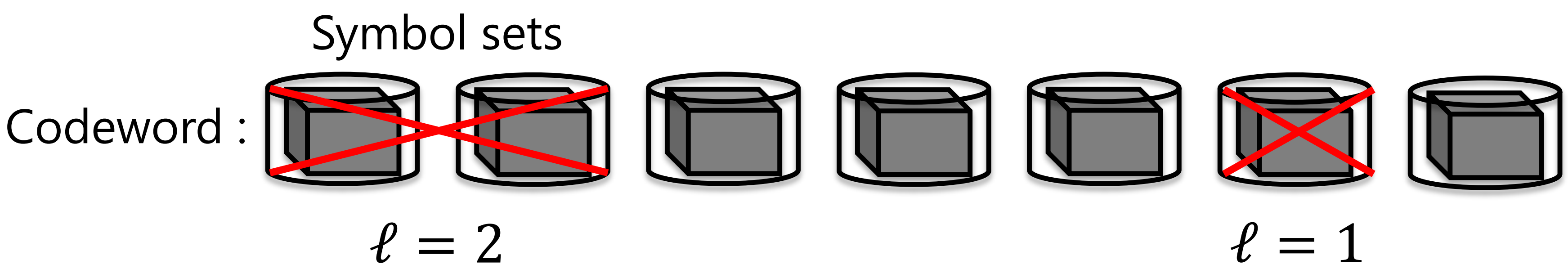
ℓ-locality (r_ℓ) [Rawat et al. 14]
locality for ℓ symbols repair (single value of ℓ)



Motivation

- Generalized Definition of locality

Joint locality $(r_1, r_2, \ldots)$ [Kim et al. 15]
a set of localities for ℓ symbols repair (multiple values of ℓ)



Motivation

- Generalized Definition of locality

Joint locality $(r_1, r_2, \ldots)$ [Kim et al. 15]
a set of localities for ℓ symbols repair (multiple values of ℓ)

Which one is better? C_1? or C_2?

$$G_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \quad G_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

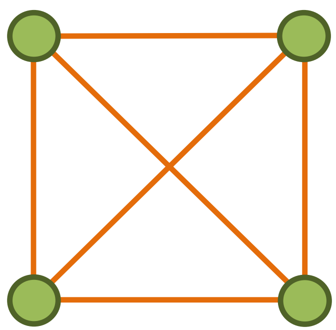
$r_1 = 2 \quad < \quad r_1 = 3$

$r_2 = 5 \quad > \quad r_2 = 4$

Motivation

- Construction of BLRCs with good joint locality
  - Graph-based construction [Kim et al. 15]

Ex)



$$G = \begin{pmatrix} 1000 & 111000 \\ 0100 & 100110 \\ 0010 & 010101 \\ 0001 & 001011 \end{pmatrix}$$

Joint locality : $(r_1, r_2, r_3) = (2, 3, 4)$

Any bounds ?

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Alphabet-dependent Bounds

Theorem 1 [Proposed]

Let $\mathcal{C}$ be an $(n,k,d)_q$ code with joint locality $\{r_l | 1 \le l \le d-1\}$. Then $\mathcal{C}$ satisfies

$$k \le \min_{\boldsymbol{l}_{[z]}} \left[J_z + k_{opt}^{(q)}(n - I_z, d) \right],$$

where

- z is a positive integer such that $\sum_{i=1}^{z} r_{l_i} < k$,
- $\boldsymbol{l}_{[z]} = \{l_i | 1 \le l_i \le d-1, 1 \le i \le z\}$,
- $J_z = \sum_{i=1}^{z} r_{l_i}$,
- $I_z = \sum_{i=1}^{z} (r_{l_i} + l_i)$.

Alphabet-dependent Bounds

Theorem 1 [Proposed]

≠ Cadambe et. al.'s bound

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Alphabet-dependent Bounds

Remark 1 [Cadambe et al. 13]

Let $\mathcal{C}$ be an $(n, k, d)_q$ code with locality r. Then $\mathcal{C}$ satisfies

$$k \leq \min_{z} \left[zr + k_{opt}^{(q)}(n - z(r+1), d) \right],$$

where

- z is a positive integer such that $z \leq \left\lceil \frac{k}{r} \right\rceil - 1$ ($\Leftrightarrow zr < k$).

Joint locality ⇒ locality

Theorem 1 (proposed) ➡ Remark 1 (Cadambe et al.)

Alphabet-dependent Bounds

Theorem 1 [Proposed] - example

Consider a $[63, 6, 32]_2$ simplex code with joint locality $\{r_1 = 2, r_2 = 3, \dots\}$.

$$k \leq \min_{\boldsymbol{l}_{[z]}} \left[J_z + k_{opt}^{(2)}(n - I_z, d) \right],$$

Alphabet-dependent Bounds

Theorem 1 [Proposed] - example

Consider a $[63, 6, 32]_2$ simplex code with joint locality $\{r_1 = 2, r_2 = 3, \ldots\}$.

$$k \le \min_{\boldsymbol{l}_{[z]}} \left[J_z + k_{opt}^{(2)}(n - I_z, d) \right],$$

Possible values $(\sum_{i=1}^{z} r_{l_i} < k = 6)$				

Alphabet-dependent Bounds

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Possible values $(\sum_{i=1}^{z} r_{l_i} < k = 6)$				
z	1	1	2	2
$\boldsymbol{l}_{[z]}$	$\{1\}$	$\{2\}$	$\{1, 1\}$	$\{1, 2\}$

Alphabet-dependent Bounds

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z	1	1	2	2
$\boldsymbol{l}_{[z]}$	$\{1\}$	$\{2\}$	$\{1,1\}$	$\{1,2\}$
J_z	2	3	4	5
I_z	3	5	6	8

Alphabet-dependent Bounds

Theorem 1 [Proposed] - example

Consider a $[63, 6, 32]_2$ simplex code with joint locality $\{r_1 = 2, r_2 = 3, \ldots\}$.

$$k \le \min_{\boldsymbol{l}_{[z]}} \left[J_z + k_{opt}^{(2)}(n - I_z, d) \right],$$

Using Plotkin bound for $k_{opt}^{(2)}(\cdot)$, we have

Possible values $(\sum_{i=1}^{z} r_{l_i} < k = 6)$				
z	1	1	2	2
$\boldsymbol{l}_{[z]}$	$\{1\}$	$\{2\}$	$\{1, 1\}$	$\{1, 2\}$
J_z	2	3	4	5
I_z	3	5	6	8

$$k \le 6$$

Alphabet-dependent Bounds

Theorem 2 [Proposed]

Let $\mathcal{C}$ be an $(n,k,d)_q$ code with joint locality $\{r_l | 1 \leq l \leq d-1\}$. Then $\mathcal{C}$ satisfies

$$d \leq \min_{\boldsymbol{l}_{[z]}} \left[d_{opt}^{(q)}(n - I_z, k - J_z) \right],$$

where

- z is a positive integer such that $\sum_{i=1}^{z} r_{l_i} < k$,
- $\boldsymbol{l}_{[z]} = \{l_i | 1 \leq l_i \leq d-1, 1 \leq i \leq z\}$,
- $J_z = \sum_{i=1}^{z} r_{l_i}$,
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Singleton-like Bounds

Corollary 1 [Proposed]

Let $\mathcal{C}$ be an $(n, k, d)_q$ code with joint locality $\{r_l | 1 \leq l \leq d-1\}$. Then $\mathcal{C}$ satisfies

$$d \leq n - k + 1 - \max_{\boldsymbol{l}_{[z]}} \sum_{i=1}^{z} l_i \,,$$

No limit to the field size

where

- z is a positive integer such that $\sum_{i=1}^{z} r_{l_i} < k$,
- $\boldsymbol{l}_{[z]} = \{l_i | 1 \leq l_i \leq d-1, 1 \leq i \leq z\}$.

Singleton-like Bounds

Corollary 1 [Proposed]

$$d \leq n - k + 1 - \max_{\boldsymbol{l}_{[z]}} \sum_{i=1}^{z} l_i$$

$l_i = l$
$r_{l_i} = r_l$

l-locality

$l_i = 1$
$r_{l_i} = r$

locality

$l_i = 1$
$r_{l_i} = k$

[Rawat et al. 14]

$$d \leq n - k + 1 - l\left(\left\lceil \frac{k}{r_l} \right\rceil - 1\right)$$

[Gopalan et al. 14]

$$d \leq n - k - \left\lceil \frac{k}{r} \right\rceil + 2$$

[Singleton bound]

$$d \leq n - k + 1$$

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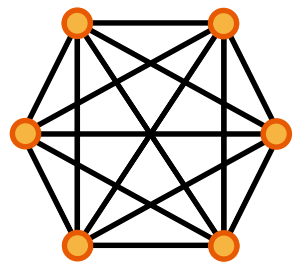
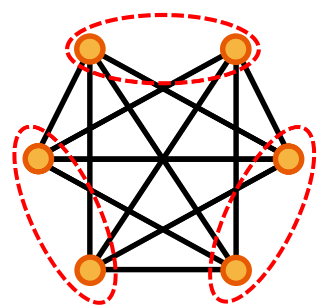
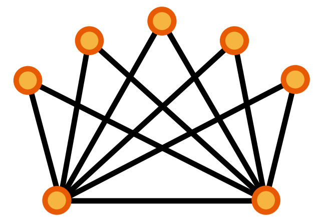
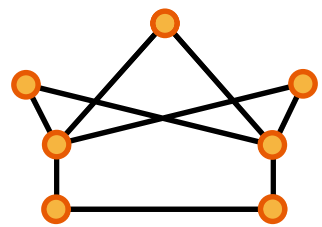
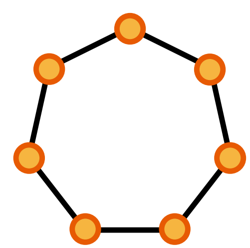
Graph-based BLRCs

Codes	Joint locality	Graph	Generator matrix
$[\frac{1}{2}k(k+1),k,k]_2$ Complete graph code [Kim et al. 15]	$(r_1,r_2)_{\text{all}} = (2,3)$ $(r_1,r_2)_{\text{info}} = (2,3)$		$\left(\begin{array}{cccccc:ccccccccccccccc} 1&0&0&0&0&0&1&1&0&1&0&0&1&0&0&0&1&0&0&0&0 \\ 0&1&0&0&0&0&1&0&1&0&1&0&0&1&0&0&0&1&0&0&0 \\ 0&0&1&0&0&0&0&1&1&0&0&1&0&0&1&0&0&0&1&0&0 \\ 0&0&0&1&0&0&0&0&0&1&1&1&0&0&0&1&0&0&0&1&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0&1 \\ 0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1 \end{array}\right)$
$[\frac{1}{2}k\left(k-\frac{k}{p}+2\right),k,k-\frac{k}{p}+1]_2$ Complete multipartite graph code [Kim et al. 15]	$(r_1,r_2)_{\text{all}} = (2,4)$ $(r_1,r_2)_{\text{info}} = (2,3)$		$\left(\begin{array}{cccccc:cccccccccccc} 1&0&0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0 \\ \hdashline 0&0&1&0&0&0&1&0&0&0&1&0&0&0&1&1&0&0 \\ 0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&0&1&1 \\ \hdashline 0&0&0&0&1&0&0&0&1&0&0&0&1&0&1&0&1&0 \\ 0&0&0&0&0&1&0&0&0&1&0&0&0&1&0&1&0&1 \end{array}\right)$

Graph-based BLRCs

Codes	Joint locality	Graph	Generator matrix
$[3k-3,k,3]_2$ Tiara code [Proposed]	$(r_1,r_2)_{\text{all}} = (2,3)$ $(r_1,r_2)_{\text{info}} = (2,3)$		$\left(\begin{array}{ccccccc\|ccccccccccc} 1&0&0&0&0&0&0&1&1&0&1&0&1&0&1&0&1&0\\ 0&1&0&0&0&0&0&1&0&1&0&1&0&1&0&1&0&1\\ 0&0&1&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&1&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0&1&1&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&1&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&1 \end{array}\right)$
$[3k-5,k,3]_2$ Crown code [Kim et al. 15]	$(r_1,r_2)_{\text{all}} = (2,4)$ $(r_1,r_2)_{\text{info}} = (2,3)$		$\left(\begin{array}{ccccccc\|ccccccccc} 1&0&0&0&0&0&0&1&0&0&0&1&1&0&1&0\\ 0&1&0&0&0&0&0&1&1&0&0&0&0&1&0&1\\ 0&0&1&0&0&0&0&0&1&1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&1&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&1&1&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0&1&1&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0&0&0&1&1 \end{array}\right)$
$[3k-3,k,3]_2$ Ring code [Kim et al. 15]	$(r_1,r_2)_{\text{all}} = (2,4)$ $(r_1,r_2)_{\text{info}} = (2,4)$		$\left(\begin{array}{ccccccc\|ccccccc} 1&0&0&0&0&0&0&1&0&0&0&0&0&1\\ 0&1&0&0&0&0&0&1&1&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&1&1&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&1&1&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&1&1&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&1&1&0\\ 0&0&0&0&0&0&1&0&0&0&0&0&1&1 \end{array}\right)$

Graph-based BLRCs

Some optimal and almost optimal codes

Graph-based BLRCs	Theorem 1 [Proposed]	Theorem 2 [Proposed]	Remark 1 [Cadambe et al.]
$[10,4,4]_2$ CG code	√	√	a.o.
$[8,4,3]_2$ CMG code	√	√	√
$[9,4,3]_2$ Tiara code	√	a.o.	a.o.
$[10,5,3]_2$ Crown code	√	√	√
$[6,3,3]_2$ Ring code	√	√	√
$[2^k-1,k,2^{k-1}]_2$ Simplex code	√	√	√

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Conclusions

- Two alphabet-dependent bounds for LRCs with joint locality are proposed
- New graph-based BLRCs (Tiara codes) with good joint locality and high rate are proposed
- Some optimal and almost optimal codes with certain choice of parameters are found
- Optimal code construction with general parameters is open