Hypergraph-Based Binary Locally Repairable Codes With Availability

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***Abstract*— We study a hypergraph-based code construction for binary locally repairable codes (LRCs) with availability. A symbol of a code is said to have (r, t)-availability if it can be recovered from t disjoint repair sets of other symbols, each set of size at most r. We refer a systematic code to an LRC with $(r, t)_i$-availability if its information symbols have (r, t)-availability and a code to an LRC with $(r, t)_a$-availability if its all symbols have (r, t)-availability. We construct binary LRCs with $(r, t)_i$-availability from linear r-uniform t-regular hypergraphs. As a special case, we also construct binary LRCs with $(r, t)_a$-availability from labeled linear r-uniform t-regular hypergraphs. Moreover, we extend the hypergraph-based codes to increase the minimum distance. All the proposed codes achieve a well-known Singleton-like bound with equality.**

***Index Terms*— Distributed storage systems, locally repairable codes, hypergraphs, graph-based codes.**

I. Introduction

In THE *Big Data* era, due to the dramatically increase of data, distributed storage systems are becoming more and more important. To ensure reliability against storage node failures, various coding techniques [1]–[3] have been employed in the systems. The simplest and most commonly used way is replication, but it is extremely inefficient in storage space, equipments, devices, and cost for powering and cooling. On the opposing side, maximum distance separable (MDS) codes have minimal storage overhead for a given reliability requirement, but suffer from inefficiency in repair process.

Motivated by the desire to reduce repair cost of codes for distributed storage systems, an interesting notion, locality, was introduced in the literature [4]. More precisely, if a symbol of a code $\mathcal{C}$ can be expressed as a function of at most r other symbols in $\mathcal{C}$, the symbol is said to have locality r. We call codes having the property locally repairable codes (LRCs) [5].

In addition to the locality, another important property of LRCs is the availability [6]. A symbol of a code $\mathcal{C}$ is said to have (r, t)-availability if it can be recovered from t disjoint repair sets of other symbols, each set of size at most r. We refer a systematic code to an LRC with $(r, t)_i$-availability if its information symbols have (r, t)-availability and a code to an LRC with $(r, t)_a$-availability if its all symbols have (r, t)-availability. An LRC with $(r, t)_a$-availability can tolerate multiple node failures up to t failures in any local repair process. Moreover, such LRCs with (r, t)-availability ensure $t + 1$ parallel reads for each symbol, which is appealing in distributed storage systems containing so-called *hot data* that is frequently and simultaneously accessed by many users.

For LRCs with $(r, t)_i$-availability, an upper bound on the minimum distance was presented in [6] under the following condition. Let $\mathcal{C}$ be an $[n, k, d]_q$ LRC. Every information symbol of $\mathcal{C}$ has t disjoint repair sets, each set of size at most r, such that any repair set contains only one parity symbol. Then, the minimum distance d of $\mathcal{C}$ is bounded by

$$d \leqslant n - k - \left\lceil \frac{kt}{r} \right\rceil + t + 1. \quad (1)$$

A construction of codes which achieve the bound (1) with equality is also given in [6]. The key of the construction is how to design suitable binary matrix, called membership matrix, to obtain additional local parities from original global parities of a systematic MDS code. Rawat *et al.* [6] proposed two explicit designs of the membership matrix based on resolvable designs [7] and zigzag codes [8]. Some other designs of the membership matrix are studied in [9]. From the results of [6], we can obtain Remark 1.

Remark 1: There are four classes of binary LRCs achieving the bound (1) *as the following. Here,* p *is a positive integer.*

1) *For* $r \mid kt$, $[k + \frac{kt}{r}, k, t + 1]_2$ *codes.*
2) *For* $r \mid kt$, $[k + \frac{kt}{r} + p, k, t + 1 + p]_2$ *codes.*
3) *For* $r \nmid kt$, $[k + \lceil \frac{kt}{r} \rceil, k, t + 1]_2$ *codes.*
4) *For* $r \nmid kt$, $[k + \lceil \frac{kt}{r} \rceil + p, k, t + 1 + p]_2$ *codes.*

Recently, Hao *et al.* [10] found all the binary LRCs meeting the bound (1) for $t = 1$. Hao and Xia [11] showed that regular low-density parity-check (LDPC) codes with girth greater than 4 can be used to construct binary LRCs for the class 1). They proposed three constructions from finite geometry LDPC codes [12] and array LDPC codes [13]. Based on the same framework, Su [14] proposed two constructions from circulant permutation matrices and affine permutation matrices. It is also noted that, since good binary LDPC codes are constructed based on combinatorial designs, finite fields, and finite geometries, binary LRCs attaining the bound (1) have connections with these fields. Wang *et al.* [15] proposed binary LRCs with $(r, t)_a$-availability. We can easily check that the codes are a subset of the class 1). For $t \geqslant 2$, to the best of our knowledge, any general construction of binary LRCs for the class 2) is not known yet, and the existence of binary LRCs for the classes 3) and 4) is open.

In this letter, we focus on binary LRCs for the classes 1) and 2) for $r, t \geqslant 2$. First, we present a new code construction utilizing a connection between binary LRC design and the proper hypergraph design. Based on the construction, we construct binary LRCs with $(r, t)_i$-availability from linear r-uniform hypergraphs [16] with vertices of degree at least t. In particular, we construct binary LRCs for the class 1) from linear r-uniform t-regular hypergraphs [16]. We also construct

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binary LRCs with $(r, t)_a$-availability using specially designed hypergraphs with $v = \binom{r+t-1}{t}$ vertices. Moreover, we extend binary LRCs for the class 1) to binary LRCs for the class 2).

The rest of this letter is organized as follows. In Section II, we first provide a connection between binary LRCs and hypergraphs. Then, we propose hypergraph-based binary LRCs with $(r, t)_i$-availability and with $(r, t)_a$-availability. In Section III, we propose extended hypergraph-based binary LRCs. In Section IV, we conclude the letter.

II. Hypergraph-Based Binary LRCs

In [17] and [18], the authors constructed binary LRCs from simple graphs [16]. We generalize the construction using linear uniform hypergraphs [16]. A hypergraph is a generalization of a graph in which an edge can join any number of vertices. A simple hypergraph is an unweighted, undirected hypergraph containing no loops or multiple edges. A hypergraph is linear if it is simple and every two edges shares at most one vertex. A r-uniform hypergraph is a hypergraph such that every edge contains r vertices. We first define a hypergraph code and determine the code parameters and properties such as the code length, code rate, minimum distance, locality, and availability.

Definition 1: Consider a linear r-uniform hypergraph $\mathcal{H}$. In $\mathcal{H}$, the vertices are associated with information symbols, and the edges are associated with parity symbols. Each parity symbol is calculated by binary addition of information symbols contained in the parity symbol. Then, we can convert $\mathcal{H}$ into a code $\mathcal{C}$. We call this code $\mathcal{C}$ a r-uniform hypergraph code.

Consider a linear r-uniform hypergraph $\mathcal{H}$ with v vertices and e edges. Then, the corresponding code $\mathcal{C}$ has the dimension v and length $v + e$. The number of edges e can be represented by using the degree of vertices. We denote by τ_i the degree of i-th vertex in ascending order, i.e. $\tau_1 \leqslant \tau_2 \leqslant \ldots \leqslant \tau_v$. Then, $e = \frac{1}{r}\sum_{i=1}^{v} \tau_i$, and thus, the code length of $\mathcal{C}$ is $v + \frac{1}{r}\sum_{i=1}^{v} \tau_i$.

Lemma 1: Let $\mathcal{C}$ be a $[v + \frac{1}{r}\sum_{i=1}^{v} \tau_i, v, d]_2$ r-uniform hypergraph code, and G be a systematic generator matrix of $\mathcal{C}$. For any s rows in G, we denote by $\mathbf{x}$ a row of them and by $W(s)$ the weight of a row obtained as a result of summing up all the s rows by column-wise binary addition. Then, $W(s)$ is greater than or equal to the weight of $\mathbf{x}$.

Proof: Using column and row permutation, we can always make G to the form G' in Fig. 1. In G', the top s rows are the selected rows, and the first row is $\mathbf{x}$. Consider a submatrix consisting of the remaining $s-1$ rows except $\mathbf{x}$. We can divide the submatrix into M_1 and M_0 as shown in the figure. Then, $W(s)$ can be written as follows:

$$W(s) = \begin{cases} wt(\mathbf{x}), & \text{for } s = 1, \\ wt(\mathbf{x}) - W_1(s-1) + W_0(s-1), & \text{for } 2 \leqslant s \leqslant v, \end{cases}$$

where $wt(\mathbf{x})$ is the weight of $\mathbf{x}$. $W_1(s-1)$ and $W_0(s-1)$ are the numbers of columns of odd weight in M_1 and M_0, respectively. Since M_1 contains only disjoint non-zero columns and at least one all-zero column, $W_1(s-1) \leqslant s-1$. Since M_0 contains at least $s-1$ columns of weight one, $W_0(s-1) \geqslant s-1$. Thus, we have $W(s) \geqslant wt(\mathbf{x})$. ■

Theorem 1: Let $\mathcal{C}$ be a $[v + \frac{1}{r}\sum_{i=1}^{v} \tau_i, v, d]_2$ r-uniform hypergraph code. Then, the minimum distance d of $\mathcal{C}$ is $\tau_1 + 1$.

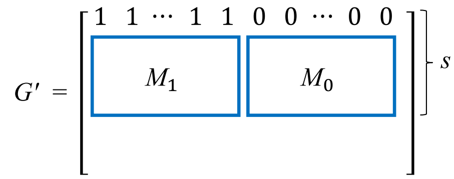
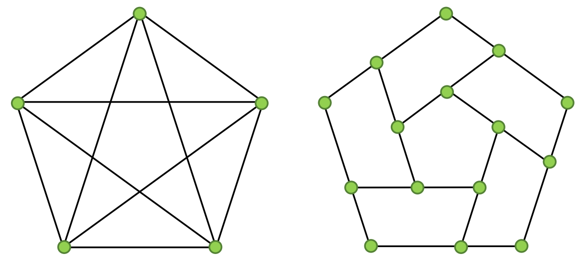


Fig. 1. A permuted generator matrix of the hypergraph code.



(a) (b)

$$G = \begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&1
\end{pmatrix}.$$

(c)

Fig. 2. (r, t)-hypergraphs. (a) A $(2, 4)$-hypergraph; (b) A $(3, 2)$-hypergraph; (c) A generator matrix of the code corresponding to (b).

Proof: Consider a systematic generator matrix G of $\mathcal{C}$. By Lemma 1, the minimum distance d of $\mathcal{C}$ is the minimum weight of a row in G. Consider a hypergraph $\mathcal{H}$ corresponding to G. The weight of a row in G is one more than the degree of corresponding vertex in $\mathcal{H}$. Thus, we have $d = \tau_1 + 1$. ■

Theorem 2: Let $\mathcal{C}$ be a $[v + \frac{1}{r}\sum_{i=1}^{v} \tau_i, v, d]_2$ r-uniform hypergraph code. Then, $\mathcal{C}$ has $(r, \tau_1)_i$-availability.

Proof: Let $\mathcal{H}$ be a linear r-uniform hypergraph corresponding to $\mathcal{C}$. Every vertex in $\mathcal{H}$ is connected with at least τ_1 edges. Every edge contains r vertices, and every two edges shares at most one vertex. Thus, each information symbol of $\mathcal{C}$ has at least τ_1 disjoint repair sets of cardinality r. ■

A. Hypergraph-Based Binary LRCs With $(r, t)_i$-Availability

We first consider linear r-uniform t-regular hypergraphs such that every edge contains r vertices, and every vertex is contained in t edges. In the rest of this letter, we refer to a linear r-uniform t-regular hypergraph as an (r, t)-hypergraph. Figure 2(a) and 2(b) show two examples of (r, t)-hypergraphs. Figure 2(c) is a generator matrix of the code corresponding to Fig. 2(b).

We note that, for any positive integers r and t, the necessary conditions for the existence of an (r, t)-hypergraph on v vertices are $v \geqslant t(r-1)+1$ and $r \mid vt$. For $r = 2$ and 3, the necessary conditions are also sufficient. For $r = 4$, no non-existence result of an (r, t)-hypergraph on v vertices is known under the necessary conditions. For $r \geqslant 5$, the necessary conditions are not sufficient. The detailed information of existence under the necessary conditions is presented in [7]. It is also proved that an (r, t)-hypergraph on v vertices always exists with sufficiently large v [19]. One good example is

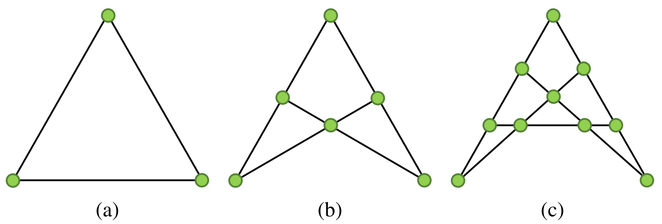


Fig. 3. (r, t)-tower hypergraphs. (a) A (2, 2)-tower hypergraph; (b) A (3, 2)-tower hypergraph; (c) A (4, 2)-tower hypergraph.

$v = \binom{r+t-1}{t}$. To obtain (r, t)-hypergraphs, we can also use equivalence relations and useful properties in Remark 2.

Remark 2: [7] *Consider an (r, t)-hypergraph $\mathcal{H}$ with v vertices and e edges. Then, the followings are satisfied:*

1) *$\mathcal{H}$ is a (v_t, e_r)-configuration.*
2) *$\mathcal{H}$ such that $\lambda(v-1) = t(r-1)$ for a positive integer λ is a (v, e, t, r, λ)-design.*
3) *$\mathcal{H}$ corresponds to a biregular graph with v vertices of degree t, e vertices of degree r, and the girth at least 6. We call this graph the Levi graph of $\mathcal{H}$.*
4) *The dual of $\mathcal{H}$ is a (t, r)-hypergraph with e vertices and v edges.*
5) *If we have an (r, t)-hypergraph $\mathcal{H}'$ with v' vertices and e' edges, with $\mathcal{H}$ and $\mathcal{H}'$ being disjoint, then we can obtain a new (r, t)-hypergraph $\mathcal{H}''$ with $v'' = v + v'$ vertices and $e'' = e + e'$ edges by combining $\mathcal{H}$ and $\mathcal{H}'$.*

Definition 2: Consider an (r, t)-hypergraph $\mathcal{H}$ with v vertices and $\frac{vt}{r}$ edges. In $\mathcal{H}$, the vertices are associated with information symbols, and the edges are associated with parity symbols. Each parity symbol is calculated by binary addition of information symbols contained in the parity symbol. Then, we can convert $\mathcal{H}$ into a code $\mathcal{C}$. We call this code $\mathcal{C}$ an (r, t)-hypergraph code.

We note that a $[v + \frac{vt}{r}, v, d]_2$ (r, t)-hypergraph code has the minimum distance $d = t + 1$ by Theorem 1 and $(r, t)_i$-availability by Theorem 2. With the parameters, we can easily check that (r, t)-hypergraph codes achieve the bound (1).

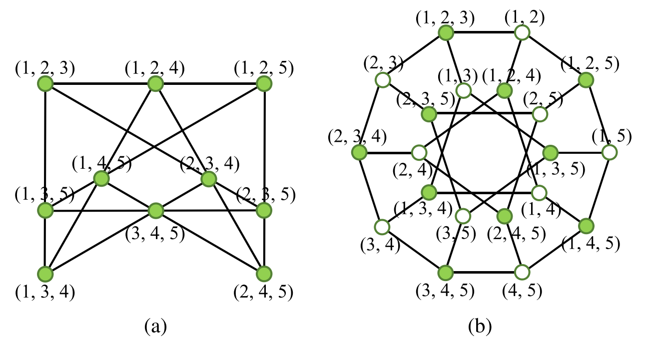
Remark 3: Using the (2, 4)*-hypergraph in* Fig. 2(a)*, we can obtain a* $[15, 5, 5]_2$ *LRC with* $(2, 4)_i$*-availability. It can not be obtained from any three constructions in* [11] *or two constructions in* [14]*. Moreover, all the codes in* [11] *and* [14] *are obtained from our construction, and hence, ours is more general and encompasses those in* [11] *and* [14]*.*

B. Hypergraph-Based Binary LRCs With $(r, t)_a$-Availability

To obtain binary LRCs with $(r, t)_a$-availability, we first define specially designed hypergraphs using vertex labelling.

Definition 3: For given two positive integers r and t, consider a hypergraph $\mathcal{H}$ with v vertices labelled using t-subsets of a $(r + t - 1)$-set, that is, $v = \binom{r+t-1}{t}$. Connect a subset of vertices with an edge if and only if all the vertices in the subset share $t-1$ indices, and label the edge using the common indices. We call this hypergraph $\mathcal{H}$ an (r, t)-tower hypergraph.

Figure 3 shows (r, t)-tower hypergraphs for $r = 2, 3, 4$ and $t = 2$ without the labels. Each of them has single tower structure. A tower forms a dependent set for edges contained in the tower. That is, each edge in the tower can be represented by the other edges. For $t \geqslant 3$, an (r, t)-tower hypergraph $\mathcal{H}$ contains multiple tower structures. That is, each edge in $\mathcal{H}$ is contained in $t - 1$ towers which are disjoint except for that edge. Figure 4 shows a (3, 3)-tower hypergraph, its Levi graph representation, and a corresponding generator matrix.



$$G = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0
\end{pmatrix}.$$

(c)

Fig. 4. A (3, 3)-tower hypergraph with labels. (a) A (3, 3)-tower hypergraph; (b) A Levi graph representation of (a); (c) A generator matrix of the code corresponding to (a).

Definition 4: Consider an (r, t)-tower hypergraph $\mathcal{H}$. In $\mathcal{H}$, the vertices are associated with information symbols, and the edges are associated with parity symbols. Each parity symbol is calculated by binary addition of information symbols contained in the parity symbol. Then, we can convert $\mathcal{H}$ into a code $\mathcal{C}$. We call this code $\mathcal{C}$ an (r, t)-tower hypergraph code.

Consider a (r, t)-tower hypergraph $\mathcal{H}$ with v vertices and e edges. In $\mathcal{H}$, each edge contains r vertices, and each vertex is contained in t disjoint edges. Thus, $\mathcal{H}$ is also an (r, t)-hypergraph. Since $v = \binom{r+t-1}{t}$ and $e = \frac{vt}{r} = \binom{r+t-1}{r}$, the corresponding (r, t)-tower hypergraph code has the code length $\binom{r+t}{t}$, code rate $\frac{r}{r+t}$, and minimum distance $t + 1$.

Theorem 3: Let $\mathcal{C}$ be an (r, t)-tower hypergraph code. Then, $\mathcal{C}$ has $(r, t)_a$-availability.

Proof: Since $\mathcal{C}$ is also an (r, t)-hypergraph code, $\mathcal{C}$ has $(r, t)_i$-availability. Now, we will show every parity symbol also has (r, t)-availability. Consider a tower hypergraph $\mathcal{H}$ corresponding to $\mathcal{C}$ and an edge e_i in $\mathcal{H}$. For a $(t-2)$-subset of $t-1$ indices of e_i, the number of edges containing the $t-2$ indices is $r + 1$, including e_i. We denote by F the edge set. Every vertex contained in any edge in F is shared by only two edges in F. Thus, all the edges in F form a dependent set, that is, e_i can be recovered by the other r edges in F. The number of such sets is $\binom{t-1}{t-2} = t - 1$, and all the sets share only e_i. Since e_i can be also recovered by r vertices contained in e_i, e_i has total t disjoint repair sets of size r. ■

From the proof of Theorem 3, we can expect that, when a parity symbol is lost, a repair set is selected from the indices of vertices and edges in the corresponding tower hypergraph. For example, if an edge with $t - 1$ indices is lost, we can choose a set of edges which share $t - 2$ indices among $t - 1$ indices of the edge. We can also choose a set of all the vertices which contain $t - 1$ indices of the edge.

TABLE I

HYPERGRAPH CODES AND THEIR POSSIBLE EXTENSIONS

$[n, k, d]_2$ hypergraph codes	$(r, t)_i$-availability	Possible p
[6, 3, 3], [8, 4, 3], [10, 5, 3], [12, 6, 3], [14, 7, 3]	(2, 2)	1
[10, 6, 3], [15, 9, 3]	(3, 2)	1
[15, 10, 3], [18, 12, 3]	(4, 2)	1
[18, 6, 5]	(2, 4)	1
[15, 5, 5], [30, 10, 5]	(2, 4)	1, 2, 3
[28, 7, 7]	(2, 6)	1, 2, 3, 4, 5
[16, 8, 4], [18, 9, 4], [20, 10, 4]	(3, 3)	1, 2

III. EXTENDED HYPERGRAPH-BASED BINARY LRCs

We try to find some extensions which increase the minimum distance of hypergraph codes and also make the extended codes achieve the bound (1) with equality. Let $\mathcal{C}$ be an $[n, k, d]_2$ (r, t)-hypergraph code and G be the generator matrix of $\mathcal{C}$. We consider three cases: (1) d is odd. (2) k is odd and $r = 2, t = k - 1$. (3) $k \geqslant 8, 7 \nmid k$ and $r = t = 3$. For the first case, we add a column of all ones to G. For the second case, we add p columns of all ones to G. Here, p is a positive integer less than or equal to $k - 2$. For the third case, we add one column or two columns of all ones to G.

In the extension for case (1), since each row of G has odd weight, the minimum distance increases by one. To check the extension for case (2), recall the definition of $W(s)$ in Lemma 1. From the structure of G, $W(s) = s + s(k - s)$, and thus, $W(s)$ has the smallest value when $s = 1, k$ and the second smallest value when $s = 2, k-1$. Since $W(2)-W(1) = k - 2$, the minimum distance increases by p, $1 \leqslant p \leqslant k - 2$. Similarly, in the extension for case (3), from the structure of G, it is easy to see that $W(s)$ has the smallest value 4 when $s = 1$ and the second smallest value $\geqslant 6$ when $s = 2$. Thus, the minimum distance increases by one or two.

Based on the extensions, we call the extended code with p columns of all ones a p-extended hypergraph code. If there are two extended hypergraph codes, $[n_1, k_1, d]_2$ p_1-extended (r, t)-hypergraph code and $[n_2, k_2, d]_2$ p_2-extended (r, t)-hypergraph code, using Remark 2-5), we can also obtain $[n_1 + n_2, k_1 + k_2, d]_2$ p-extended (r, t)-hypergraph codes for any positive integer p, $1 \leqslant p \leqslant \min(p_1, p_2)$.

We confirmed that the extensions work well by a computer simulation. Table I shows some examples of them.

IV. CONCLUSIONS

We proposed a hypergraph-based code construction for binary LRCs with availability. From the construction, if we have an (r, t)-hypergraph, we can construct a binary LRC with $(r, t)_i$-availability. The necessary conditions for the existence of an (r, t)-hypergraph were presented. As a special case, we can construct binary LRCs with $(r, t)_a$-availability from (r, t)-tower hypergraphs. For given r and t, an (r, t)-tower hypergraph always exists. We also proposed extended hypergraph codes to increase the minimum distance. Interestingly, all the proposed codes attain the bound (1) with equality. As a future work, one may obtain binary LRCs with more various parameters using new equivalence relations and useful graph properties. The existence problem of two remaining classes in Remark 1 for $r, t \geqslant 2$ is also an interesting topic.

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