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The Existence of Circular Florentine Arrays

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Dedicated to Professor Solomon W. Golomb on his 60^{th} birthday

Abstract—A $k \times n$ circular Florentine array is an array of n distinct symbols in k *circular* rows such that

(1) each row contains every symbol exactly once, and
(2) for any pair of distinct symbols (a, b) and for any integer m from 1 to $n-1$ there is *at most* one row in which b occurs m steps to the right of a.

For each positive integer $n = 2, 3, 4, \ldots$, define $F_c(n)$ to be the *maximum* number such that an $F_c(n) \times n$ circular Florentine array exists.

From the main construction of this paper for a set of mutually orthogonal Latin squares (MOLS) having an additional property, and from the known results on the existence/nonexistence of such MOLS obtained by others, it is now possible to reduce the gap between the upper and lower bounds on $F_c(n)$ for infinitely many additional values of n not previously covered. This is summarized in the table for all odd n up to 81.

1. INTRODUCTION

It would always be better to begin by a few examples rather than a formal definition to describe a combinatorial object called *circular Florentine array*. An example of a 4×5 circular Florentine array is shown in Figure 1. Two other examples are shown in Figures 2 and 3, which are 4×15 and 4×27 circular Florentine arrays, respectively. Note that each row has every symbol $0, 1, \ldots, n-1$ exactly once. Observe further that for any symbol a and for any integer $m = 1, 2, \ldots, n-1$, the symbols in m steps circularly to the right of a are all distinct throughout the array.

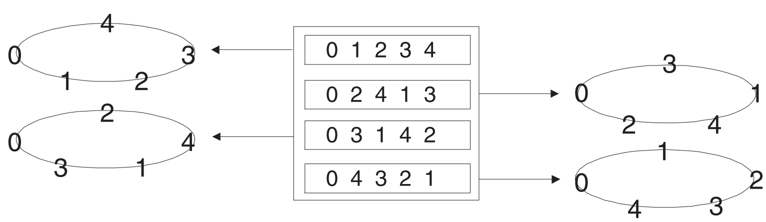

Figure 1. A 4×5 circular Florentine array.

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0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	7	1	8	2	12	3	11	9	4	13	5	14	6	10
0	4	11	7	10	1	13	9	5	8	3	6	2	14	12
0	13	7	2	11	6	14	10	3	5	12	9	1	4	8

Figure 2. A 4×15 circular Florentine array.

14	15	16	17	18	19	20	21	22	23	24	25	26	0	1	2	3	4	5	6	7	8	9	10	11	12	13
21	7	10	22	9	23	8	24	11	25	14	26	12	0	15	1	13	2	16	3	19	4	18	5	17	20	6
18	24	15	7	5	25	13	16	6	8	26	17	23	0	4	10	1	19	21	11	14	2	22	20	12	3	9
10	18	22	6	3	1	15	19	2	13	23	11	7	0	20	16	4	14	25	8	12	26	24	21	5	9	17

Figure 3. A 4×27 circular Florentine array.

Formally, a $k \times n$ circular Florentine array is an array of n distinct symbols in k circular rows such that each row contains every symbol exactly once and that for any pair of distinct symbols (a, b) and for any integer m from 1 to $n-1$ there is *at most* one row in which b occurs m steps (circularly) to the right of a. For convenience, define $F_c(n)$ for each positive integer n to be *the maximum number such that an $F_c(n) \times n$ circular Florentine array exists*. The examples shown in Figures 1–3 prove that $F_c(5) \geq 4$, $F_c(15) \geq 4$, and $F_c(27) \geq 4$.

Proposition 1.1. *$p - 1 \leq F_c(n) \leq n - 1$, for each $n = 2, 3, 4, \ldots$, where p is the smallest prime factor of n.*

Proof. Let $n \geq 2$ be a positive integer. For any fixed symbol a, since there are at most $n-1$ ordered pairs of the form (a, x) where $a \neq x$, the number of circular Florentine rows that could possibly exist is clearly at most $n-1$. On the other hand, it is not hard to show that the top $p-1$ rows of the multiplication table mod n with borders in the natural order form a $(p-1) \times n$ circular Florentine array, where p is the smallest prime factor of n. ∎

The exact value of $F_c(n)$ and the related problems have been investigated by others [1–4] for the direct application of $F_c(n)$ rows of a circular Florentine array into communication signal designs such as frequency hopping patterns, radar arrays, and sonar arrays. There are at least two previous results concerning the value of $F_c(n)$. These are

(1) $F_c(n) = 1$ whenever n is even [1], and
(2) $F_c(n) \leq n - 2$ whenever "Bruck-Ryser Theorem" rules out the existence of a finite projective plane of order n [1,5–7].

In Section 2, we will prove the following necessary and sufficient condition for the existence of a $k \times n$ circular Florentine array. The construction in the proof results in not only the above two previous results, but also some refinement for the exact value of $F_c(n)$ for infinitely many values of n other than listed in (1) or (2) above.

Theorem 1.1. *There exists a circular Florentine array of size $k \times n$ if and only if there exists a set of k mutually orthogonal Latin squares of order n such that the rows of any square are cyclic shifts of each other and that every square is obtainable from any other only by permuting the rows.*

Finally, all possible values of $F_c(n)$ for $3 \leq n \leq 81$, n odd, are shown in Table 1. This summarizes our current state of knowledge on $F_c(n)$ and is an updated table from [3].

2. PROOF BY CONSTRUCTION AND ITS IMPLICATION

Proof of Theorem 1.1. Suppose we are given a $k \times n$ circular Florentine array, which will be denoted by $C = (c(i, j))$ in matrix notation where $c(i, j) \in \{a_0, a_1, \ldots, a_{n-1}\}$ for $i = 1, 2, \ldots, k$

Table 1. Possible values of $F_c(n)$ for all odd n from 3 to 81.

n	$F_c(n)$	LB	UB	n	$F_c(n)$	LB	UB
3	2	n is *prime*		43	42	n is *prime*	
5	4	n is *prime*		45	$2,\ldots,43$	$*$	Corollary 2.3
7	6	n is *prime*		47	46	n is *prime*	
9	2	search		49	$6,\ldots,48$	$*$	$\star$
11	10	n is *prime*		51	$2,\ldots,48$	$*$	Corollary 2.4
13	12	n is *prime*		53	52	n is *prime*	
15	4	search		55	$4,\ldots,54$	$*$	$\star$
17	16	n is *prime*		57	$7,\ldots,55$	‡	Corollary 2.1
19	18	n is *prime*		59	58	n is *prime*	
21	$5,\ldots,19$	‡	Corollary 2.1	61	60	n is *prime*	
23	22	n is *prime*		63	$6,\ldots,62$	‡	$\star$
25	$4,\ldots,24$	$*$	$\star$	65	$4,\ldots,63$	$*$	Corollary 2.3
27	$4,\ldots,26$	search	$\star$	67	66	n is *prime*	
29	28	n is *prime*		69	$2,\ldots,66$	$*$	Corollary 2.4
31	30	n is *prime*		71	70	n is *prime*	
33	$3,\ldots,30$	†	Corollary 2.4	73	72	n is *prime*	
35	$4,\ldots,33$	$*$	Corollary 2.3	75	$2,\ldots,73$	$*$	Corollary 2.3
37	36	n is *prime*		77	$6,\ldots,75$	$*$	Corollary 2.1
39	$3,\ldots,38$	†	$\star$	79	78	n is *prime*	
41	40	n is *prime*		81	$2,\ldots,80$	$*$	$\star$

$*$	Basic lower bound, one less than the smallest prime factor.
†	Theorem 1.1 and [8].
‡	Theorem 1.1 and [9] (see Section 3 for †, ‡, and "search").
$\star$	Basic upper bound, which is $n-1$.
Cor.	See Section 2 for Corollaries.

and $j = 0, 1, 2, \ldots, n-1$. Assume that the top row is in the natural order $a_0, a_1, a_2, \ldots, a_{n-1}$ (rename the symbols if necessary).

We will construct a set of k squares, $L_1, L_2, \ldots, L_k$, of size $n \times n$ using only the cyclic shifts of $a_0, a_1, a_2, \ldots, a_{n-1}$. Therefore, it is sufficient to specify the left-most column of each square (column 0). Rows and columns of the square have labels $0, 1, 2, \ldots, n-1$. For each $x = 1, 2, \ldots, k$, consider the following relation:

$$\text{for } i = 0, 1, 2, \ldots, n-1, \qquad c(x,i) = a_j \quad \Longrightarrow \quad L_x(j,0) = a_i. \tag{2.1}$$

First, note that the left-most column of L_x given by equation (2.1) is the inverse permutation of those induced by the row x of C. Here, we use the interpretation of each row as a permutation of symbols by the rule $c(1,i) \to c(x,i)$ for $i = 0, 1, 2, \ldots, n-1$. Therefore, each column of L_x is a permutation. Since each row is a cyclic shift of $a_0, a_1, \ldots, a_{n-1}$, this proves that L_x is Latin.

To show the orthogonality of L_s and L_t for some $1 \leq s < t \leq k$, suppose, on the contrary, that they are not orthogonal. Then, there are two corresponding positions in both the squares such that the two ordered pairs from these positions are the same. That is, for some indices x, y, u, and v,

$$L_s(x,y) = L_s(u,v) = a_i \qquad \text{and} \qquad L_t(x,y) = L_t(u,v) = a_j,$$

for some symbol a_i and a_j. This implies

$$\begin{aligned} L_s(x,0) &= a_{i\ominus y}, \qquad L_t(x,0) = a_{j\ominus y}, \qquad \text{and} \\ L_s(u,0) &= a_{i\ominus v}, \qquad L_t(u,0) = a_{j\ominus v}, \end{aligned}$$

where $\ominus$ denotes mod n subtraction. This can happen only if

$$\begin{aligned} c(s, i\ominus y) &= a_x = c(t, j\ominus y), \qquad \text{and} \\ c(s, i\ominus v) &= a_u = c(t, j\ominus v). \end{aligned}$$

But, it implies that the symbol a_u is $y \ominus v$ steps to the right of a_x in both the row s and the row t of C, a desired contradiction.

Similarly for the converse. ∎

From the above theorem, the following two results can easily be derived.

Corollary 2.1. *(See [1,5,6].)* $F_c(n) \leq n-2$ *whenever the Bruck-Ryser Theorem rules out the existence of a finite projective plane of order* n*, or more specifically, whenever* $n \equiv 1$ *or* $2 \pmod 4$ *such that the square-free part of* n *contains at least one prime factor* p *which is congruent to* 3 mod 4.

Corollary 2.2. *(See [1].)* $F_c(n) = 1$ *whenever* n *is even.*

Proof. Note that any of the Latin squares given by the construction is essentially an addition table of integers mod n, and hence does not have a single *transversal* [10] if n is even. ∎

Additional results on the nonexistence of an $(n-1)\times n$ circular Florentine array can be obtained from the nonexistence of MOLS described in Theorem 1.1 by de Launey [11,12]. This can be translated in our terminology as the following corollary.

Corollary 2.3. $F_c(n) \leq n-2$ *whenever the existence of the set of* $n-1$ *MOLS of order* n *having the property described in Theorem 1.1 is ruled out, or more specifically, whenever* m *is a quadratic nonresidue* mod p *where* $m \not\equiv 0 \pmod p$ *is an integer dividing the square-free part of* n *and* $p \neq 2$ *is a prime divisor of* n.

For example, for each positive integer t, if $n = 5^t \cdot 7$, then $n \equiv 3 \pmod 4$ and $7 \equiv 2 \pmod 5$ is a quadratic nonresidue modulo 5. Therefore, there does not exist an $(n-1)\times n$ circular Florentine array whenever $n = 5^t \cdot 7$ for any positive integer t. These are infinitely many additional values of n, not covered by the Bruck-Ryser Theorem (see Corollary 2.1).

Woodcock [13] in 1986 proved independently that the set of $n-1$ MOLS of order n having the property described in Theorem 1.1 does not exist whenever $n \equiv 15 \pmod{18}$. Though these values of n are already ruled out by Corollary 2.3, the proof actually rules out the existence of $n-2$ such squares.

Corollary 2.4. $F_c(n) \leq n-3$ *whenever* $n \equiv 15 \pmod{18}$.

3. LOWER BOUND ON $F_c(n)$ AND CONCLUSION

The basic lower bound which is one less than the smallest prime factor of n (Proposition 1.1) can be improved by the constructions from Jungnickel [9] and Theorem 1.1. For $n < 100$, this gives $F_c(21) \geq 4$, $F_c(57) \geq 7$, and $F_c(63) \geq 6$. Nazarok [14] found 5 MOLS of order 21 in which every row of a square is a cyclic shift of its top row. This gives $F_c(21) \geq 5$.

Schellenberg, van Rees, and Vanstone in 1978 have searched by computer for those MOLS described in Theorem 1.1 [8]. From their explicit examples of 3 MOLS of order $n = 33$ and $n = 39$, and from Theorem 1.1, we have $F_c(33) \geq 3$ and $F_c(39) \geq 3$.

It is believed that an $(n-1) \times n$ circular Florentine array does not exist whenever n is not a prime. When p is a prime, the multiplication table of the integers $1, 2, \ldots, p-1 \bmod p$ (by adjoining a constant column of all 0s) provides an example of a $(p-1) \times p$ circular Florentine array. Therefore, $F_c(p) = p - 1$ if p is a prime. In addition to the corollaries in the previous section, two more cases were determined by some exhaustive computer search, which are $F_c(9) \leq 2$, and $F_c(15) \leq 4$, the latter by Wilson and Roth [15].

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