The CRC Handbook
Of
Combinatorial Designs
(Incomplete; Partly Edited)

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1 Tuscan Squares

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1.1 Definitions, Examples and Enumeration

An $r \times n$ *Tuscan-k rectangle* has r rows and n columns such that (1) each row is a permutation of the n symbols and that (2) for any two distinct symbols a and b, and for each m from 1 to k, there is *at most* one row in which b is m steps to the right of a.

A *Tuscan rectangle* is a Tuscan-1 rectangle. When $r = n$, it is a *Tuscan square* of order n. A Tuscan square is in *standard form* when the top row and the left-most column contain the symbols in natural order. A *Roman* square is both Tuscan and latin, and was originally called a *row complete latin square*. A Tuscan-$(n-1)$ rectangle is a *Florentine* rectangle, and is a *Vatican* rectangle when it is also latin.

1.1 Remark Clearly, if there exists an $r \times n$ Tuscan rectangle, then necessarily $r \leq n$.

1.2 Example Florentine squares of orders 2, 4, and 6. A Tuscan square of order 7 that is not Tuscan-2. A Tuscan-2 square of order 8 that is not Tuscan-3.

1	2
2	1

1	2	3	4
2	4	1	3
3	1	4	2
4	3	2	1

1	2	3	4	5	6
2	4	6	1	3	5
3	6	2	5	1	4
4	1	5	2	6	3
5	3	1	6	4	2
6	5	4	3	2	1

6	1	5	2	4	3	7
2	6	3	5	4	7	1
5	7	2	3	1	4	6
4	2	5	1	6	7	3
3	6	2	1	7	4	5
1	3	2	7	5	6	4
7	6	5	3	4	1	2

1	2	3	4	5	6	7	8
2	1	6	8	7	3	5	4
3	2	7	1	8	4	6	5
4	1	7	5	3	8	6	2
5	8	1	4	7	2	6	3
6	1	5	2	4	8	3	7
7	4	2	8	5	1	3	6
8	2	5	7	6	4	3	1

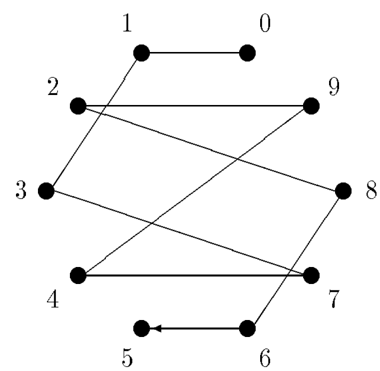
1.3 Table The number of *standard form* Tuscan-k squares of order n. [9, 8]

$k \backslash n$	2	3	4	5	6	7	8	9	10	11	12	13
1	1	0	1	0	736	466144	$\geq 3 \cdot 10^7$	$\geq 3 \cdot 10^7$	≥ 72	≥ 1	≥ 964	≥ 1
2		0	1	0	1	0	6	?	≥ 2	?	≥ 2	?
3			1	0	1	0	0	0	≥ 1	?	≥ 1	?
4				0	1	0	0	0	≥ 1	?	≥ 1	?
5					1	0	0	0	≥ 1	?	≥ 1	?
6						0	0	0	≥ 1	?	≥ 1	?
7							0	0	≥ 1	?	≥ 1	?
8								0	1	?	≥ 1	?
9									1	0	≥ 1	?
10										0	≥ 1	?
11											≥ 1	?
12												?

A *circular Tuscan-k array* of size $r \times n$ satisfies the same condition as a Tuscan-k rectangle with r *circular* rows containing n symbols.

1.4 Remark Adjoining a column of asterisks to the left of an $n \times n$ Tuscan square makes it an $n \times (n+1)$ circular Tuscan array. This works since the left-most and right-most columns must be permutations. Rotating the rows so that another symbol fills the left-most column, and then deleting it, gives a new $n \times n$ Tuscan square. This is the *add-asterisk transformation.*

1.5 Remark A *polygonal-path* (PP) Tuscan-k square has columns which are cyclic shifts of each other. If it becomes a circular Tuscan-k array when a column of asterisks is adjoined at the far-left, it is a *polygonal-path* circular Tuscan-k array.



1.6 Example Successive rotation of the symmetric polygonal-path, 0 1 3 7 4 9 2 8 6 5, shown produces a partition of K^*_{10} into 10 Hamiltonian directed paths. When written as 10 rows, this becomes a polygonal-path Tuscan-2 square which is not Tuscan-3. [8] For the general case, see Theorem 1.15(2).

1.2 Existence and Related Combinatorial Objects

1.7 Theorem If n is even, there exists an $n \times n$ Roman square (row complete latin square).

1.8 Construction To construct a Roman square of order $n = 2m$, use the "zig-zag" polygonal-path, namely, $0, 1, 2m-1, 2, 2m-2, \ldots, m+1, m$, and its successive rotations.

1.9 Theorems (existence)

1. If $n > 0$, $n \neq 3, 5$, then there exists a Tuscan square of order n. [12]
2. For every integer $n > 5$, there exists a non-latin Tuscan square of order n all of whose add-asterisk transforms are non-latin. [6]
3. If p is a prime and either $p \equiv 7 \pmod{12}$ or $p \equiv 5 \pmod{24}$, then there exists a Tuscan-2 square of order $2p$ whose transpose is also a Tuscan-2 square. [1]
4. If p is the smallest prime factor of n, then $C(x,y) \equiv xy \pmod{n}$ for $x = 1, 2, \ldots, p-1$ and $y = 0, 1, \ldots, n-1$ will construct a $(p-1) \times n$ circular Florentine array (CFA).
5. If n is even, there does not exist a $2 \times n$ CFA. [8]
6. A PP CFA of size $n \times (n+1)$ exists if and only if $n+1$ is an odd prime. [7]

1.10 Remark There exists a Roman square of order pq whenever $p < q$ are odd primes and 2 is a primitive root mod p. (See Theorem 1.15(1) and Theorem IV.??.) Roman squares of odd order n are known for $n \in \{9, 15, 21, 27, 39, 55, 57\}$. [2] Roman squares of orders 3, 5 and 7 do not exist.

1.11 Remark It is not known whether any non-latin Florentine squares exist.

1.12 Remark Two Tuscan-k rectangles are *isomorphic* if their standard forms differ by (1) a substitution of symbols, (2) permuting the rows, and/or (3) taking the reverse order of symbols in every row. Every known Florentine square is isomorphic to the one from Theorem 1.9(4) with n prime.

1.13 Remark A symmetric PP circular Tuscan-2 array of size $n \times (n+1)$ is known for every even n such that $8 \le n \le 50$. [8]

1.14 Remark For $n \le 20$, every PP Tuscan-3 square of order n is Vatican. For $n \le 28$ except for $n = 22$, every symmetric PP Tuscan-3 square is Vatican. For $n = 22$, there exists a *unique* symmetric PP Tuscan-3 square which is not Tuscan-4. Three *inequivalent* symmetric PP Tuscan-3 squares (not Tuscan-4) are known for $n = 30$. [8]

1.15 Theorems (connections to other combinatorial objects)

1. If a group of order n is sequenceable, there exists a Tuscan square of order n whose transpose is also Tuscan (a *complete latin square*). (See §IV.??.)
2. A Tuscan square of order n is equivalent to a decomposition of the complete directed graph with n vertices into n Hamiltonian directed paths. [6, 12]
3. There exists a PP Vatican square of order n if and only if there exists a singly periodic Costas array of order n. [7] (See §IV.??.)
4. An $n \times (n+1)$ circular Tuscan-k array induces k MOLS of order $n+1$. [7]
5. An $r \times n$ circular Florentine array induces r MOLS, $L_1, L_2, \ldots, L_r$, of order n. To construct these, first rotate the rows of the CFA so that the leftmost symbol in every row is the same. Then let the top row of L_x be row x of the CFA for each x from 1 to r. The remaining rows of each latin square are obtained by using cyclic shifts of its top row such that (1) L_1 has a constant diagonal and (2) L_x for $x \ne 1$ has as its diagonal, the top row of L_1. [10]
6. An $r \times n$ circular Florentine array is equivalent to an $(n, r+1; 1)$ difference matrix over the cyclic group of order n. [10] (See §IV.??.)
7. A pair of orthogonal latin squares cannot both be Vatican. [10]

1.3 Florentine Rectangles and Circular Florentine Arrays

1.16 Examples [10] The two non-isomorphic 6×7 Florentine rectangles having a constant middle column which are not from Theorem 1.9(4). The two non-isomorphic 8×9 Florentine rectangles having a constant middle column.

1	2	3	4	5	6	7
2	5	7	4	3	1	6
3	7	1	4	6	5	2
5	3	6	4	7	2	1
6	1	5	4	2	7	3
7	6	2	4	1	3	5

1	2	3	4	5	6	7
2	6	1	4	7	5	3
3	1	7	4	6	2	5
5	7	6	4	1	3	2
6	5	2	4	3	7	1
7	3	5	4	2	1	6

1	2	3	4	5	6	7	8	9
2	1	7	9	5	8	3	6	4
3	7	2	8	5	4	1	9	6
4	9	8	1	5	7	6	3	2
6	8	4	7	5	2	9	1	3
7	3	1	6	5	9	2	4	8
8	6	9	3	5	1	4	2	7
9	4	6	2	5	3	8	7	1

1	2	3	4	5	6	7	8	9
2	9	6	1	5	8	3	7	4
3	6	9	7	5	4	2	1	8
4	1	7	9	5	3	8	6	2
6	8	4	3	5	1	9	2	7
7	3	2	8	5	9	1	4	6
8	7	1	6	5	2	4	9	3
9	4	8	2	5	7	6	3	1

$F(n)$ = the maximum such that an $F(n) \times n$ Florentine rectangle exists.
$F_c(n)$ = the maximum such that an $F_c(n) \times n$ circular Florentine array exists.

1.17 Table Possible values of $F_c(n)$ and $F(n)$ [10, 11].

n	$F_c(n)$	n	$F_c(n)$
1	1	21	5,...,19
3	2	23	22
5	4	25	4,...,24
7	6	27	4,...,26
9	2	29	28
11	10	31	30
13	12	33	3,...,30
15	4	35	4,...,33
17	16	37	36
19	18	39	3,...,38

n	$F(n)$	n	$F(n)$	n	$F(n)$
1	1	11	10	21	7,...,21
2	2	12	12	22	22
3	2	13	12,13	23	22,23
4	4	14	7,...,14	24	6,...,24
5	4	15	7,...,15	25	6,...,25
6	6	16	16	26	6,...,26
7	6	17	16,17	27	6,...,27
8	7	18	18	28	28
9	8	19	18,19	29	28,29
10	10	20	6,...,20	30	30

1.18 Example A 4×27 circular Florentine array. [11]

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
0	4	10	1	19	21	11	14	2	22	20	12	3	9	18	24	15	7	5	25	13	16	6	8	26	17	23
0	15	1	13	2	16	3	19	4	18	5	17	20	6	21	7	10	22	9	23	8	24	11	25	14	26	12
0	20	16	4	14	25	8	12	26	24	21	5	9	17	10	18	22	6	3	1	15	19	2	13	23	11	7

1.4 Applications of Tuscan Squares

1.19 Remark In a sequence of visual impressions one flash card may have some effect on the impression given by the next. This bias can be cancelled by using n sequences corresponding to the rows of an $n \times n$ Tuscan-1 square. [3]

1.20 Remark In Frequency-Hopped multiple-access communication systems any two users should avoid hopping simultaneously to the same carrier frequency, an event termed a *hit*. Assigning a row of an $r \times n$ circular Florentine array to each user as a frequency hopping sequence minimizes the number of hits between any two users. [10, 11]

1.5 See Also

[4]	A discussion of "horizontally complete" (i.e. Roman) squares can be found on pages 299-302.
[5]	Chapter 3 deals with row complete latin squares and sequencings of groups

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