

## Generalized Locality for Distributed Storage Codes

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Friday 1/30 – Saturday 1/31 2015 KICS-NA IT Workshop

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#### Large-Scale Storage System

- Warehouse-scale data center
- Thousands of servers, Petabytes of disc space



Google Data Center

http://www.google.com/about/datacenters/

At *Facebook*, it is quite typical to have 20 or more node failures per day. (Sathiamoorthy-12)



- Codes for Distributed Storage Systems
  - Store data while protecting against node failures



- Coding+Network → New open problems
  - Minimize Storage Size
  - Minimize Repair Bandwidth
  - Minimize the number of nodes connected for repair



#### System model

 $\alpha$  : Size of a storage node



## **Distributed Storage Codes**

- Minimum Storage Regenerating (MSR) Codes
  - Interference Alignment method (Shah-12)



## **Distributed Storage Codes**

- Minimum Bandwidth Regenerating (MBR) Codes
  - Repair-by-Transfer method (Shum-12)



## **Distributed Storage Codes**

- Locally Repairable Codes (LRC)
  - Minimize the number of nodes accessed for repairing a node



#### **Concept of Locality**

(Gopalan-12) introduced codes with locality & minimum distance bound

(Papailiopoulos-12) generalized the minimum distance bound for non-linear codes and vector codes

#### (Tamo-14, Prakash-14, Rawat-14) devoted the locality in handling of

multiple node failures

A. S. Rawat, A. Mazumdar, and S.Vishwanath. (2014). "Cooperative local repair in distributed storage," [Online]. Available: http://arxiv.org/abs/1409.3900.

(This presentation) introduces generalized locality and related bounds

#### **Construction of Codes with Locality**

(Huang-07) Pyramid codes

(Kamath-I3) MSR/MBR-local codes

(Huang-12) Local reconstruction codes

(Sathiamoorthy-I3) Locally repairable codes

(Krishnan-I4, Shahabinejad-I4) Codes with locality for Hadoop

(This presentation) proposes a construction of codes with GOOD generalized locality

#### Preliminary

Let  $G : \mathbb{F}_q^M \mapsto \mathbb{F}_q^{n \cdot \alpha}$  be an encoding (generator) function:

$$G(\boldsymbol{X}) = \boldsymbol{Y} = (Y_1, Y_2, \dots, Y_n),$$

where  $X \in \mathbb{F}_q^M$  and  $Y \in \mathbb{F}_q^{n \cdot \alpha}$ .

We denote the code determined by the encoding function G as an  $(n, (M, \alpha), d)_q$  code C, where d is the minimum distance.



Motivation

To construct distributed storage codes,

We should consider all possible failure patterns together.



Motivation

Assume that the code is optimized **only** for **2-erasures** 



#### Definition

Consider an  $(n, (M, \alpha), d)_q$  code C given by the above. Let  $\ell$  be an integer with  $1 \leq \ell \leq d - 1$ , and  $E \subset [n]$  with  $|E| = \ell$ . We denote by  $Y_E$  the set of coded symbols  $Y_i, i \in E$ . That is  $Y_E = \{Y_i | i \in E, |E| = \ell\}$ . Then,

1)  $R(E) \subseteq [n] \setminus E$  is called a repair set for  $Y_E$  if every  $Y_i, i \in E$  can be regenerated by a set of functions on  $Y_j, j \in R(E)$ . (Rawat-14) Note that there can be many different repair sets for a given set of symbols  $Y_E$ .

2) The integer r is called the locality of  $Y_E$  if r is the minimum of all the cardinalities of the repair sets for  $Y_E$ .

It is called the  $\ell$ -locality of  $\mathcal{C}$  if every set of symbols  $Y_E$ ,  $|E| = \ell$ , has the locality at most r. The  $\ell$ -locality of  $\mathcal{C}$  is denoted by  $r_{\ell}$ . (Rawat-14)

3) The set of integers,  $r_1, r_2, ..., r_{d-1}$ , is called the generalized locality of C.

Example



A repair set of  $Y_1$  :  $R(1) = \{3,4\}$ A repair set of  $Y_2$  :  $R(2) = \{4,5,6\}$ 

Let  $E = \{1,2\}$ . A repair set of  $Y_E : R(E) = \{3,4,5,6\}$ 

Locality of  $Y_1$  : 2 Locality of  $Y_2$  : 3 Locality of  $Y_E$  : 4

Locality (1-locality) of C : 3 2-locality of C : 4 Generalized Locality of C : ( $r_1 = 3, r_2 = 4$ )

#### **Bounds for Codes with GL**

#### Theorem

Let C be an  $(n, (M, \alpha), d)_q$  code with generalized locality  $(r_1, r_2, ..., r_{\ell})$ . Then, the minimum distance of C is bounded as

$$d \leq \min_{\ell \geq 1} \left( n - \left[ \frac{M}{\alpha} \right] + 1 - \ell \cdot \left( \left[ \frac{M}{r_{\ell} \cdot \alpha} \right] - 1 \right) \right).$$

	Target Codes			Locality Types			
	$\begin{array}{l} Scalar \\ (\alpha = 1) \end{array}$	Vector $(\alpha \ge 1)$	Linear	Non- linear	$\begin{array}{l} 1 \text{-locality} \\ (\ell = 1) \end{array}$	$\begin{array}{l} \ell \text{-locality} \\ (\ell \geq 1) \end{array}$	Generalized locality
Gopalan-12	0		0		0		
Forbes-13	0		0	0	0		
Papailiopoulos-12	0	0	0	0	0		
Rawat-14	0		0	0	0	0	
Ours	Ο	Ο	Ο	Ο	Ο	Ο	Ο

## Simpler version (linear and scalar) and some derived new bounds

Let C be an  $[n, k, d]_q$  code with generalized locality  $(r_1, r_2, ..., r_\ell, )$ .

$$d \leq \min_{\ell \geq 1} \left( n - k + 1 - \ell \cdot \left( \left\lceil \frac{k}{r_{\ell}} \right\rceil - 1 \right) \right)$$

and, hence,

$$R(\mathcal{C}) \le \min_{\ell \ge 1} \left( \frac{r_{\ell}}{r_{\ell} + \ell} \right)$$

and

$$\left[\frac{k \cdot \ell}{n-k+1-d+\ell}\right] \leq r_\ell \leq k \ \text{ for } 1 \leq \ell \leq d-1$$

**Example:** For  $[7,3,4]_2$  code and  $\ell = 3$ ,

$$3 = \left[\frac{3 \cdot 3}{7 - 3 + 1 - 4 + 3}\right] \le r_{\ell} \le 3$$
 implies  $r_3 = 3$ 

## Construction of codes with $(r_1, r_2) = (2, 3)$

• One choice would be simplex codes with the parameter  $n = 2^k - 1, k, d = 2^{k-1}$ .

• Examples:

• 
$$G = \begin{pmatrix} 100 & 110 & 1 \\ 010 & 101 & 1 \\ 001 & 011 & 1 \end{pmatrix}$$
 for  $k=3$   
•  $G = \begin{pmatrix} 1000 & 111000 & 1110 & 1 \\ 0100 & 100110 & 1101 & 1 \\ 0010 & 010101 & 1011 & 1 \\ 0001 & 001011 & 0111 & 1 \end{pmatrix}$  for  $k=4$ 

 $r_{\ell} \leq \ell + 1$  (Rawat-14)

• It has  $(r_1, r_2) = (2, 3)$ .

• Code rate 
$$=\frac{k}{2^{k}-1}$$
 (VERY LOW)

• Question: Can we improve the rate maintaining the property  $(r_1, r_2) = (2, 3)$ ?

# One simple idea that works • $G = \begin{pmatrix} 1000 & 111000 \\ 0100 & 100110 \\ 0010 & 010101 \\ 0001 & 001011 \end{pmatrix}$ for k=4



- Can we prove that this modified code STILL has  $(r_1, r_2) = (2, 3)$  ?
- What is the code ? Its generator matrix has all the columns of weight 1 and weight 2 ONLY.
- It can be described as a complete graph with 4 vertices.
   Columns are all the vertices and edges of K<sub>4</sub>

#### **Complete Graph Codes**

• Construction of  $[k(k+1)/2, k, k]_2$  codes



 $K_6$  complete graph (k = 6)

Generator matrix of  $[21, 6, 6]_2$  code

**Theorem:** It has  $d_{min} = k$  and 1)  $r_1 = 2$  for  $k \ge 2$ . 2)  $r_2 = 3$  for  $k \ge 3$ . 3)  $r_{\ell} \le \min(2\ell, k)$  for  $k \ge 2$  and  $\ell \in [k - 1]$ .

#### Complete Multipartite (p-partite) Graph Codes

• Construction of  $[k(k-q+2)/2, k, k-q+1]_2$  codes



0 0 0 0 0 1 1 1 0 0 0 0 0 0

Complete 2-partite 3-uniform graph with 6 vertices (p = 2, q = 3, k = 6) Generator matrix of  $[15, 6, 4]_2$  code

**Theorem:** It has 
$$d_{min} = k - q + 1$$
 and  
1)  $r_1 = 2$  for  $k \ge 2$ .  
2)  $r_2 = \begin{cases} 3, & \text{for } q = 1 \text{ and } k \ge 3, \\ 4, & \text{for } q \ge 2 \text{ and } k \ge 3. \end{cases}$   
3)  $r_4 \le \min(2\ell, k)$  for  $k \ge 2$  and  $\ell \in [k - k]$ 

#### **Complete Multipartite Graph Codes**

Code rates

$$R = \frac{2}{k-q+2} \qquad \ge \ R_S = \frac{k}{2^{k}-1}$$

Minimum distances

$$d = k - q + 1 \quad \le d_S = 2^{k-1}$$



#### **Complete Multipartite Graph Codes**

Codes	Graphs	Generator matrices			
$[3, 2, 2]_2$ code (q = 1, k = 2)	(Complete graph)	$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$			
$[6, 3, 3]_2$ code (q = 1, k = 3)	(Complete graph)	$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$			
$[10, 4, 4]_2$ code (q = 1, k = 4)	(Complete graph)	$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$			
[21, 6, 6] <sub>2</sub> code (q = 1, k = 6)	(Complete graph)	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0$			

#### **Complete Multipartite Graph Codes**



#### **Simplex Codes**

Let k be a positive integer,  $n = 2^k - 1$ , and let G be a  $k \times n$  matrix whose columns are all the distinct non-zero vectors of  $\mathbb{F}_2^k$ . Let C be an  $[n, k, d]_2$  code that has G as its generator matrix. Then, C is called a binary simplex code with  $d = 2^{k-1}$ .

$$G = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

A generator matrix of  $[7,3,4]_2$  simplex code

Number of symbol sets with cardinality $\ell$ and locality $r$	<i>r</i> = 1	r=2	r = 3	$\ell$ -locality of $\mathcal C$
$\ell = 1$		n		<i>r</i> <sub>1</sub> = 2
$\ell = 2$			$\binom{n}{2}$	$r_2 = 3$
$\ell = 3$			$\binom{n}{3}$	r <sub>3</sub> = 3

#### **Identity Repeated (IR) Simplex Codes**

Let G be a  $k \times n$  matrix constructed by adding an identity matrix in front of a generator matrix of a simplex code. Let C be an  $[n, k, d]_2$  code that has G as its generator matrix. Then, C is called an IR-simplex code with  $n = 2^k - 1 + k$  and  $d = 2^{k-1} + 1$ .

$$G = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

A generator matrix of  $[10,3,5]_2$  IR-simplex code

Number of symbol sets with cardinality $\ell$ and locality $r$	<i>r</i> = 1	r=2	r = 3	$\ell$ -locality of $\mathcal C$
$\ell = 1$	2 <i>k</i>	n-2k		<i>r</i> <sub>1</sub> = 2
$\ell=2$		k(2n-2k-1)	$\binom{n-2k}{2}$	<i>r</i> <sub>2</sub> = 3
$\ell = 3$		4k(k-1)	$\binom{n}{3} - 4k(k-1)$	r <sub>3</sub> = 3

#### Simplex Codes & IR-Simplex Codes

#### Comparison of Localities



#### Simplex Codes & IR-Simplex Codes

#### Comparison of Localities



## Conclusion

- review distributed storage codes
- introduce "generalized locality"
- improved bounds on various parameters
- complete graph codes
- complete multipartite graph codes
- generalized locality of simplex codes
- IR-simplex codes
- some good algorithm of attaching a column to G for better locality