A generalized Milewski construction for perfect sequences

for APSK+ constellations

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2018년 부호 및 정보이론 연구회 3차 워크숍

Autocorrelation & perfect sequence

- Let $\boldsymbol{x} = \{x(n)\}_{n=0}^{L-1}$ be a complex-valued sequence of length L.

  The (periodic) autocorrelation of $\boldsymbol{x}$ at time shift τ is

$$\mathrm{C}_{\boldsymbol{x}}(\tau) = \sum_{n=0}^{L-1} x(n+\tau)x^*(n).$$

  where $n+\tau$ is computed over the integers modulo L.

- A sequence is called perfect if

$$\mathrm{C}_{\boldsymbol{x}}(\tau) = E\delta(\tau) = \begin{cases} E, & \tau \equiv 0 \pmod{L} \\ 0, & \tau \not\equiv 0 \pmod{L}. \end{cases}$$

  where $E = C_{\boldsymbol{x}}(0)$

DFT of a perfect sequence

$$\Im\{C_{\boldsymbol{x}}(\tau)\}(k) = \sum_{\tau=0}^{L-1}\left[\sum_{n=0}^{L-1} x(n+\tau)x^*(n)\right]e^{-j2\pi k\tau}$$

$$= \sum_{n=0}^{L-1}\left[\sum_{\tau=0}^{L-1} x(n+\tau)e^{-j2\pi k(n+\tau)}\right]x^*(n)e^{j2\pi kn}$$

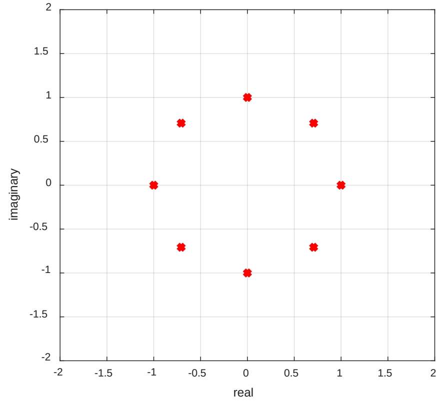
$$= X(k)\left[\sum_{n=0}^{L-1} x(n)e^{-j2\pi kn}\right]^* = X(k)X^*(k) = |X(k)|^2$$

$C_{\boldsymbol{x}}(\tau) = E\delta(\tau)$ if and only if $|X(k)|^2$ is a constant for any k.

x is a **perfect** sequence of period L **if and only if** its **DFT** has **flat magnitude**.

Constellations (alphabets)

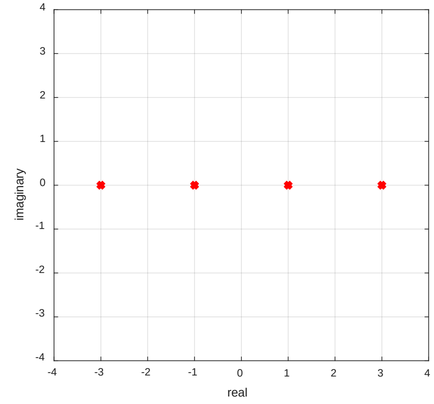
Phase shift keying (PSK) → **polyphase**



Phase only

well studied

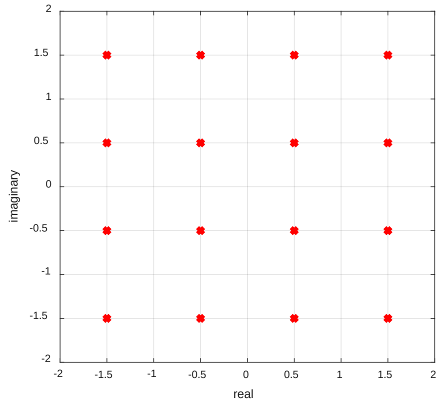
Amplitude shift keying (ASK)



real amplitude only

few results

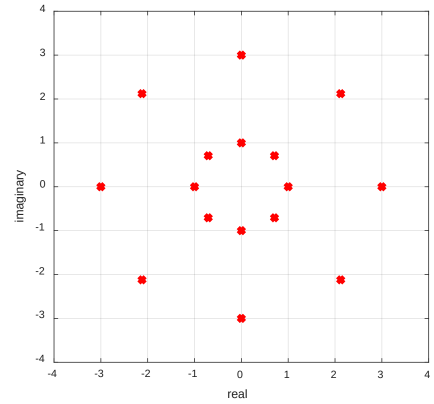
Quadrature-amplitude modulation (QAM)



Normally square

studied

Amplitude & phase shift keying (APSK)



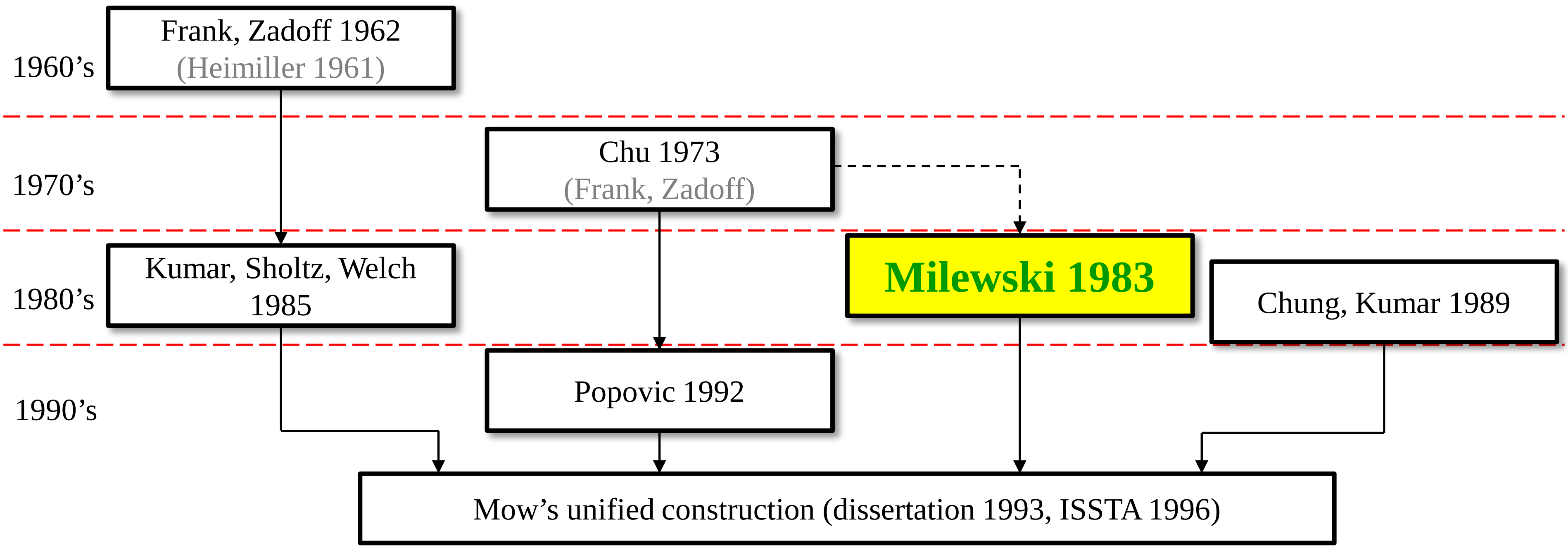
Both Phase and Amplitude

few results

Adding the zero point ⇒ PSK+, ASK+, QAM+, and APSK+

The Milewski construction

- Originally proposed to construct perfect **polyphase** sequences from those of shorter period
  - Well-known case: using the Zadoff-Chu sequence as the shorter one



- Brief history of constructing perfect polyphase sequences -

Interleaved sequence

Let $S = \{\boldsymbol{s}_0, \boldsymbol{s}_1\}$ be a set of **2** sequences of length **3**, where

$$\boldsymbol{s}_0 = \{a, b, c\}, \boldsymbol{s}_1 = \{d, e, f\}.$$

Write an array as

$$[\boldsymbol{s}_0, \boldsymbol{s}_1] = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix},$$

and then, by reading the array row-by-row,
we have

$$\boldsymbol{s} = \{a, d, b, e, c, f\}.$$

The sequence $\boldsymbol{s}$ is called an interleaved sequence.

Cyclic shift of interleaved sequence

$$\boldsymbol{s} = \{a, d, b, e, c, f\} \qquad \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$$

1 left-cyclic shift

$$\{d, b, e, c, f, a\} \qquad \begin{bmatrix} d & b \\ e & c \\ f & a \end{bmatrix}$$

1 left-cyclic shift

$$\{b, e, c, f, a, d\} \qquad \begin{bmatrix} b & e \\ c & f \\ a & d \end{bmatrix}$$

The original Milewski construction

period: $m^1 \rightarrow m^{2K+1}$

perfect polyphase sequence

$\boldsymbol{\alpha} = \{\alpha(n)\}_{n=0}^{m-1}$

A positive integer

K

Milewski Construction

Output **perfect polyphase** sequence

$\boldsymbol{s} = \{s(n)\}_{n=0}^{\boldsymbol{m}^{\boldsymbol{2K+1}}-1}$

where

$$s(n) = \alpha(q)\omega^{qr}$$

where $n = q\boldsymbol{m}^{\boldsymbol{K}} + r$.

and

$$\omega = e^{-\frac{j2\pi}{m^{1+K}}}.$$

The original Milewski construction

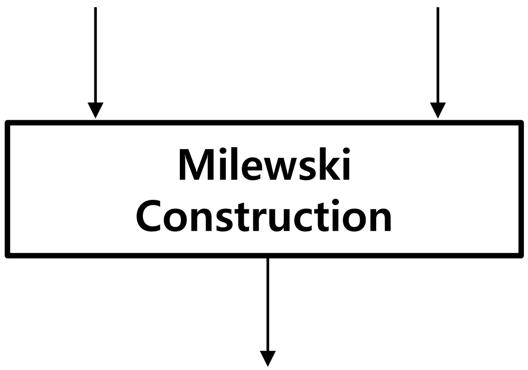
period: $m^1 \rightarrow m^{1+2K}$

perfect polyphase sequence

$\boldsymbol{\alpha} = \{\alpha(n)\}_{n=0}^{m-1}$

A positive integer

K



Output perfect polyphase sequence

$\boldsymbol{s} = \{s(n)\}_{n=0}^{m^{2K+1}-1}$

where

$$s(n) = \alpha(q)\omega^{qr}$$

where $n = q\boldsymbol{m}^{K} + r$.

and

$$\omega = e^{-\frac{j2\pi}{m^{K+1}}}.$$

$m \cdot m^K \times m^K$ array form of $\boldsymbol{s}$

Input sequence of period m

$$\begin{pmatrix}
\alpha(0) \times 1 & \alpha(0) \times 1 & \cdots & \alpha(0) \times 1 \\
\alpha(1) \times 1 & \alpha(1) \times \omega & \cdots & \alpha(1) \times \omega^{m^K-1} \\
\alpha(2) \times 1 & \alpha(2) \times \omega^2 & \cdots & \alpha(2) \times \left(\omega^{m^K-1}\right)^2 \\
\vdots & \vdots & \ddots & \vdots \\
\alpha(m-1) \times 1 & \alpha(m-1) \times \omega^{m-1} & \cdots & \alpha(m-1) \times \left(\omega^{m^K-1}\right)^{m-1} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha(0) \times 1 & \alpha(0) \times \omega^{m^K} & \cdots & \alpha(0) \times \left(\omega^{m^K-1}\right)^{m^K} \\
\alpha(1) \times 1 & \alpha(1) \times \omega^{m^K+1} & \cdots & \alpha(1) \times \left(\omega^{m^K-1}\right)^{m^K+1} \\
\alpha(2) \times 1 & \alpha(2) \times \omega^{m^K+2} & \cdots & \alpha(2) \times \left(\omega^{m^K-1}\right)^{m^K+2} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha(m-1) \times 1 & \alpha(m-1) \times \omega^{m^{K+1}-1} & \cdots & \alpha(m-1) \times \left(\omega^{m^K-1}\right)^{m^{K+1}-1}
\end{pmatrix}$$

Our generalization

perfect polyphase sequence

$\boldsymbol{\alpha} = \{\alpha(n)\}_{n=0}^{m-1}$

A positive integer

K

Milewski Construction

Output
perfect polyphase sequence

$\boldsymbol{s} = \{s(n)\}_{n=0}^{\boldsymbol{m^{2K+1}}-1}$

where

$$s(n) = \alpha(q)\omega^{qr}$$

with $n = q\boldsymbol{m^K} + r$, and

$$\omega = e^{-\frac{j2\pi}{m^{K+1}}} = e^{-\frac{j2\pi}{m(m^K)}}.$$

Perfect sequence
not necessarily polyphase

$\boldsymbol{\alpha} = \{\alpha(n)\}_{n=0}^{m-1}$

A positive integer

N

not necessarily perfect
Any polyphase sequence of length N

$\boldsymbol{\mu}$

Generalized Milewski Construction

Output
perfect sequence

$\boldsymbol{s} = \{s(n)\}_{n=0}^{\boldsymbol{mN^2}-1}$

where

$$s(n) = \mu(r)\alpha(q)\omega^{qr}$$

with $n = q\boldsymbol{N} + r$, and

$$\omega = e^{-\frac{j2\pi}{mN}}.$$

Array Form

Assume that $\boldsymbol{\mu}$ is the all-1 sequence,

Column index $r = 0, 1, 2, \dots, N-1$

Row index $q = 0,1,2,\dots,mN-1$

$$
\begin{array}{cccc}
\alpha(0) \times (\omega^{0})^{0} & \alpha(0) \times (\omega^{1})^{0} & \cdots & \alpha(0) \times (\omega^{N-1})^{0} \\
\alpha(1) \times (\omega^{0})^{1} & \alpha(1) \times (\omega^{1})^{1} & \cdots & \alpha(1) \times (\omega^{N-1})^{1} \\
\alpha(2) \times (\omega^{0})^{2} & \alpha(2) \times (\omega^{1})^{2} & \cdots & \alpha(2) \times (\omega^{N-1})^{2} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha(m-1) \times (\omega^{0})^{m-1} & \alpha(m-1) \times (\omega^{1})^{m-1} & \cdots & \alpha(m-1) \times (\omega^{N-1})^{m-1} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha(0) \times (\omega^{0})^{m(N-1)} & \alpha(0) \times (\omega^{1})^{m(N-1)} & \cdots & \alpha(0) \times (\omega^{N-1})^{m(N-1)} \\
\alpha(1) \times (\omega^{0})^{m(N-1)+1} & \alpha(1) \times (\omega^{1})^{m(N-1)+1} & \cdots & \alpha(1) \times (\omega^{N-1})^{m(N-1)+1} \\
\alpha(2) \times (\omega^{0})^{m(N-1)+2} & \alpha(2) \times (\omega^{1})^{m(N-1)+2} & \cdots & \alpha(2) \times (\omega^{N-1})^{m(N-1)+2} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha(m-1) \times (\omega^{0})^{mN-1} & \alpha(m-1) \times (\omega^{1})^{mN-1} & \cdots & \alpha(m-1) \times (\omega^{N-1})^{mN-1}
\end{array}
$$

※ $\omega = e^{-\frac{j2\pi}{mN}}$

Proof when $\boldsymbol{\mu}$ is all-1 sequence

$$\begin{pmatrix}
\alpha(0) \times (\omega^0)^0 & \alpha(0) \times (\omega^1)^0 & \cdots & \alpha(0) \times (\omega^{N-1})^0 \\
\alpha(1) \times (\omega^0)^1 & \alpha(1) \times (\omega^1)^1 & \cdots & \alpha(1) \times (\omega^{N-1})^1 \\
\alpha(2) \times (\omega^0)^2 & \alpha(2) \times (\omega^1)^2 & \cdots & \alpha(2) \times (\omega^{N-1})^2 \\
\vdots & \vdots & \ddots & \vdots \\
\alpha(m-1) \times (\omega^0)^{m-1} & \alpha(m-1) \times (\omega^1)^{m-1} & \cdots & \alpha(m-1) \times (\omega^{N-1})^{m-1} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha(0) \times (\omega^0)^{m(N-1)} & \alpha(0) \times (\omega^1)^{m(N-1)} & \cdots & \alpha(0) \times (\omega^{N-1})^{m(N-1)} \\
\alpha(1) \times (\omega^0)^{m(N-1)+1} & \alpha(1) \times (\omega^1)^{m(N-1)+1} & \cdots & \alpha(1) \times (\omega^{N-1})^{m(N-1)+1} \\
\alpha(2) \times (\omega^0)^{m(N-1)+2} & \alpha(2) \times (\omega^1)^{m(N-1)+2} & \cdots & \alpha(2) \times (\omega^{N-1})^{m(N-1)+2} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha(m-1) \times (\omega^0)^{mN-1} & \alpha(m-1) \times (\omega^1)^{mN-1} & \cdots & \alpha(m-1) \times (\omega^{N-1})^{mN-1}
\end{pmatrix} \triangleq (\boldsymbol{s}_0, \boldsymbol{s}_1, \boldsymbol{s}_2, \dots, \boldsymbol{s}_{N-1})$$

where

$$\boldsymbol{s}_r = \{s_r(q) = \alpha(q)(\omega^r)^q\}_{q=0}^{mN-1}$$

is the r-th column

Note that the DFT is linear.

Proof - continued

$$(\boldsymbol{s}_0, \boldsymbol{s}_1, \boldsymbol{s}_2, \ldots, \boldsymbol{s}_{N-1}) = \underline{(\boldsymbol{s}_0, \mathbf{0}, \mathbf{0}, \ldots, \mathbf{0})} + \cdots + (\mathbf{0}, \mathbf{0}, \ldots, \mathbf{0}, \boldsymbol{s}_{N-1})$$

$$\begin{pmatrix}
\alpha(0) \times (\omega^{0})^{\mathbf{0}} & 0 & \cdots & 0 \\
\alpha(1) \times (\omega^{0})^{\mathbf{1}} & 0 & \cdots & 0 \\
\alpha(2) \times (\omega^{0})^{\mathbf{2}} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\alpha(m-1) \times (\omega^{0})^{\boldsymbol{m-1}} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\alpha(0) \times (\omega^{0})^{\boldsymbol{m(N-1)}} & 0 & \cdots & 0 \\
\alpha(1) \times (\omega^{0})^{\boldsymbol{m(N-1)+1}} & 0 & \cdots & 0 \\
\alpha(2) \times (\omega^{0})^{\boldsymbol{m(N-1)+2}} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\alpha(m-1) \times (\omega^{0})^{\boldsymbol{mN-1}} & 0 & \cdots & 0
\end{pmatrix}$$

Proof - continued

$$(\boldsymbol{s}_0, \boldsymbol{s}_1, \boldsymbol{s}_2, \dots, \boldsymbol{s}_{N-1}) = \underline{(\boldsymbol{s}_0, \mathbf{0}, \mathbf{0}, \dots, \mathbf{0})} + \cdots + (\mathbf{0}, \mathbf{0}, \dots, \mathbf{0}, \boldsymbol{s}_{N-1})$$

$$\begin{pmatrix} \alpha(0) & 0 & \cdots & 0 \\ \alpha(1) & 0 & \cdots & 0 \\ \alpha(2) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha(m-1) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha(0) & 0 & \cdots & 0 \\ \alpha(1) & 0 & \cdots & 0 \\ \alpha(2) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha(m-1) & 0 & \cdots & 0 \end{pmatrix}$$

Proof - continued

$$(\boldsymbol{s}_0, \boldsymbol{s}_1, \boldsymbol{s}_2, \dots, \boldsymbol{s}_{N-1}) = (\boldsymbol{s}_0, \boldsymbol{0}, \boldsymbol{0}, \dots, \boldsymbol{0}) + \cdots + (\boldsymbol{0}, \boldsymbol{0}, \dots, \boldsymbol{0}, \boldsymbol{s}_{N-1})$$

$m \times N$ subarray

$$\begin{pmatrix} \alpha(0) & 0 & \cdots & 0 \\ \alpha(1) & 0 & \cdots & 0 \\ \alpha(2) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha(m-1) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha(0) & 0 & \cdots & 0 \\ \alpha(1) & 0 & \cdots & 0 \\ \alpha(2) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha(m-1) & 0 & \cdots & 0 \end{pmatrix}$$

repeated N times

Proof - continued

$m \times N$ array

$$\begin{pmatrix} \alpha(0) & 0 & \cdots & 0 \\ \alpha(1) & 0 & \cdots & 0 \\ \alpha(2) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha(m-1) & 0 & \cdots & 0 \end{pmatrix}$$

Interleaved sequence

$$\alpha(0)\,\underbrace{0\ \cdots\ 0}_{N-1\text{ times}}\ \alpha(1)\,\underbrace{0\ \cdots\ 0}_{N-1\text{ times}}\ \alpha(2)\,\underbrace{0\ \cdots\ 0}_{N-1\text{ times}}\ \cdots\ \alpha(m-1)\,\underbrace{0\ \cdots\ 0}_{N-1\text{ times}}$$

Case 1) any non-zero term meets another non-zero term

$$\alpha(\tau)\quad 0\ \cdots\ 0\quad \alpha(\tau+1)\quad 0\ \cdots\ 0\quad \alpha(\tau+2)\quad 0\ \cdots\ 0\quad \cdots\quad \alpha(\tau-1)\quad 0\ \cdots\ 0$$

$$\alpha^*(0)\quad 0\ \cdots\ 0\quad \alpha^*(1)\quad 0\ \cdots\ 0\quad \alpha^*(2)\quad 0\ \cdots\ 0\quad \cdots\quad \alpha^*(m-1)\quad 0\ \cdots\ 0$$

Case 2) any non-zero term meets zero term.

$$\alpha(\tau)\quad 0\ \cdots\ 0\quad \alpha(\tau+1)\quad 0\ \cdots\ 0\quad \alpha(\tau+2)\quad 0\ \cdots\ 0\quad \cdots\quad \alpha(\tau-1)\quad 0\ \cdots\ 0$$

$$0\quad \alpha^*(0)\quad 0\ \cdots\ 0\quad \alpha^*(1)\quad 0\ \cdots\ 0\quad \alpha^*(2)\quad 0\ \cdots\ 0\ \cdots\ 0\quad \alpha^*(m-1)\ 0\ \cdots$$

Proof - continued

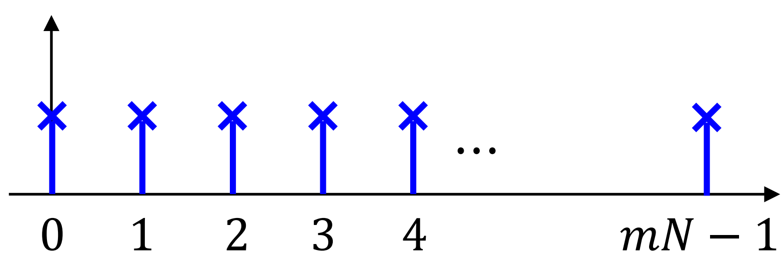
$$(\boldsymbol{s}_0, \boldsymbol{s}_1, \boldsymbol{s}_2, \dots, \boldsymbol{s}_{N-1}) = (\boldsymbol{s}_0, \mathbf{0}, \mathbf{0}, \dots, \mathbf{0}) + \cdots + (\mathbf{0}, \mathbf{0}, \dots, \mathbf{0}, \boldsymbol{s}_{N-1})$$

$$\begin{pmatrix} \alpha(0) & 0 & \cdots & 0 \\ \alpha(1) & 0 & \cdots & 0 \\ \alpha(2) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha(m-1) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha(0) & 0 & \cdots & 0 \\ \alpha(1) & 0 & \cdots & 0 \\ \alpha(2) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha(m-1) & 0 & \cdots & 0 \end{pmatrix}$$

repeated N times

A perfect sequence of period mN

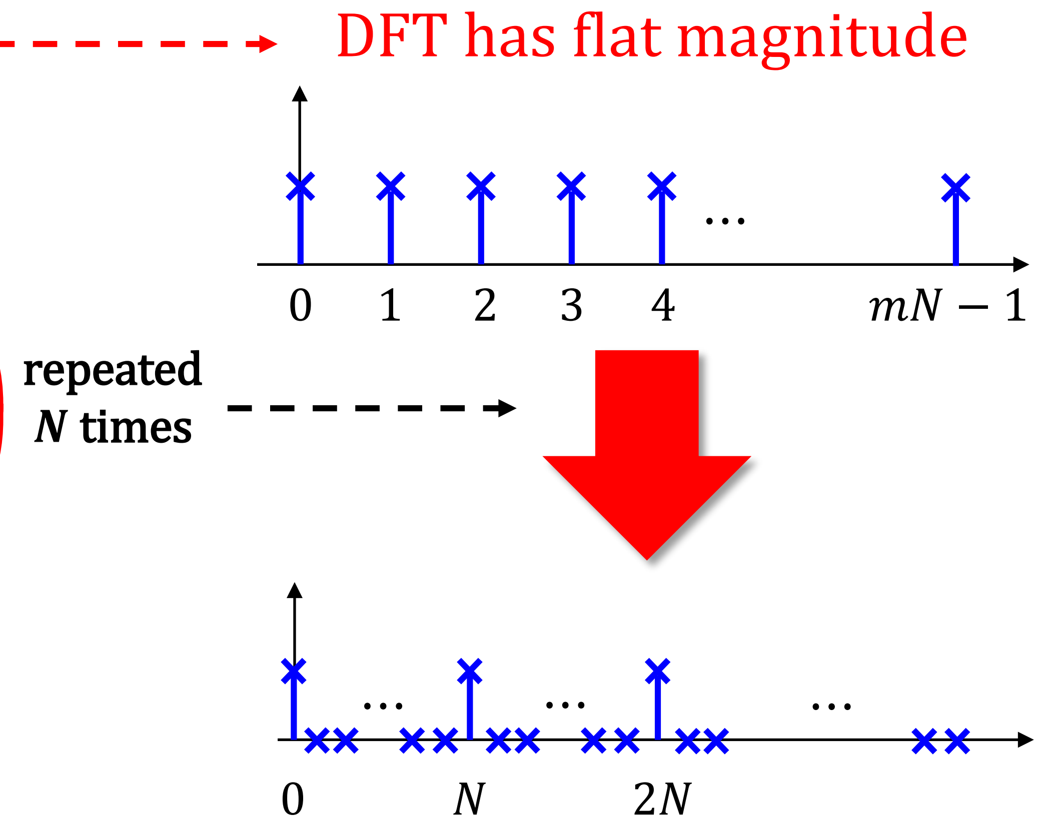
Its DFT has flat magnitude



Proof - continued

$$(\boldsymbol{s}_0, \boldsymbol{s}_1, \boldsymbol{s}_2, \ldots, \boldsymbol{s}_{N-1}) = \underline{(\boldsymbol{s}_0, \mathbf{0}, \mathbf{0}, \ldots, \mathbf{0})} + \cdots + (\mathbf{0}, \mathbf{0}, \ldots, \mathbf{0}, \boldsymbol{s}_{N-1})$$

$$\begin{pmatrix} \alpha(0) & 0 & \cdots & 0 \\ \alpha(1) & 0 & \cdots & 0 \\ \alpha(2) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha(m-1) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha(0) & 0 & \cdots & 0 \\ \alpha(1) & 0 & \cdots & 0 \\ \alpha(2) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha(m-1) & 0 & \cdots & 0 \end{pmatrix}$$



Proof - continued

$(\mathbf{0}, \boldsymbol{s}_1, \mathbf{0}, \mathbf{0}, \dots, \mathbf{0})$: 1 left-cyclic shift of $(\boldsymbol{s}_1, \mathbf{0}, \mathbf{0}, \dots, \mathbf{0})$

Cyclic shift in time domain does not affect on the magnitude in the frequency domain

Proof - continued

$(\mathbf{0}, \boldsymbol{s}_1, \mathbf{0}, \mathbf{0}, \dots, \mathbf{0})$: 1-cyclic shift of $(\boldsymbol{s}_1, \mathbf{0}, \mathbf{0}, \dots, \mathbf{0})$

Cyclic shift in time domain does not affect on magnitude at the frequency domain

$$\begin{pmatrix} \alpha(0) & 0 & \cdots & 0 \\ \alpha(1) & 0 & \cdots & 0 \\ \alpha(2) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha(m-1) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha(0) & 0 & \cdots & 0 \\ \alpha(1) & 0 & \cdots & 0 \\ \alpha(2) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha(m-1) & 0 & \cdots & 0 \end{pmatrix} \otimes \begin{pmatrix} (\omega^1)^{0} & 0 & \cdots & 0 \\ (\omega^1)^{1} & 0 & \cdots & 0 \\ (\omega^1)^{2} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (\omega^1)^{m-1} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (\omega^1)^{m(N-1)} & 0 & \cdots & 0 \\ (\omega^1)^{m(N-1)+1} & 0 & \cdots & 0 \\ (\omega^1)^{m(N-1)+2} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (\omega^1)^{mN-1} & 0 & \cdots & 0 \end{pmatrix}$$

$$(\omega^1)^x = \left(e^{-\frac{j2\pi}{mN}}\right)^x = \left(e^{-\frac{j2\pi}{mN^2}}\right)^{Nx}$$

Proof - continued

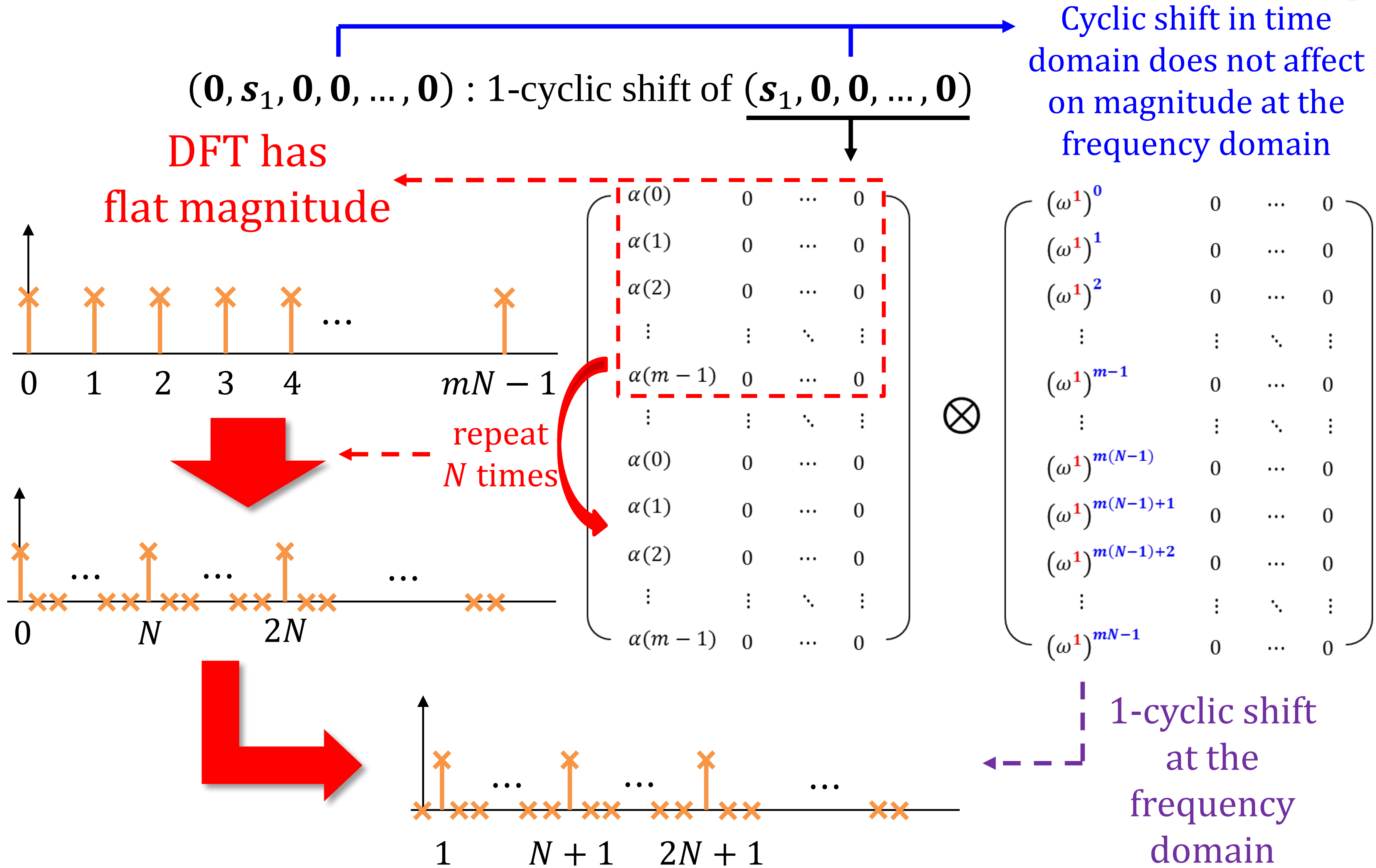
$(\mathbf{0}, \boldsymbol{s}_1, \mathbf{0}, \mathbf{0}, \dots, \mathbf{0})$: 1-cyclic shift of $(\boldsymbol{s}_1, \mathbf{0}, \mathbf{0}, \dots, \mathbf{0})$

Cyclic shift in time domain does not affect on magnitude at the frequency domain

$$
\begin{pmatrix}
\alpha(0) & 0 & \cdots & 0 \\
\alpha(1) & 0 & \cdots & 0 \\
\alpha(2) & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\alpha(m-1) & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\alpha(0) & 0 & \cdots & 0 \\
\alpha(1) & 0 & \cdots & 0 \\
\alpha(2) & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\alpha(m-1) & 0 & \cdots & 0
\end{pmatrix}
\otimes
\begin{pmatrix}
\left(e^{-\frac{j2\pi}{mN^2}}\right)^{\mathbf{0}} & \cdots & \left(e^{-\frac{j2\pi}{mN^2}}\right)^{\boldsymbol{N-1}} \\
\left(e^{-\frac{j2\pi}{mN^2}}\right)^{\boldsymbol{N}} & \cdots & \left(e^{-\frac{j2\pi}{mN^2}}\right)^{\boldsymbol{2N-1}} \\
\left(e^{-\frac{j2\pi}{mN^2}}\right)^{\boldsymbol{2N}} & \cdots & \left(e^{-\frac{j2\pi}{mN^2}}\right)^{\boldsymbol{3N-1}} \\
\vdots & & \vdots \\
\left(e^{-\frac{j2\pi}{mN^2}}\right)^{\boldsymbol{(m-1)N}} & \cdots & \left(e^{-\frac{j2\pi}{mN^2}}\right)^{\boldsymbol{mN-1}} \\
\vdots & & \vdots \\
\left(e^{-\frac{j2\pi}{mN^2}}\right)^{\boldsymbol{m(N-1)N}} & \cdots & \left(e^{-\frac{j2\pi}{mN^2}}\right)^{\boldsymbol{m(N-1)N+N-1}} \\
\left(e^{-\frac{j2\pi}{mN^2}}\right)^{\boldsymbol{[m(N-1)+1]N}} & \cdots & \left(e^{-\frac{j2\pi}{mN^2}}\right)^{\boldsymbol{[m(N-1)+1]N+N-1}} \\
\left(e^{-\frac{j2\pi}{mN^2}}\right)^{\boldsymbol{[m(N-1)+2]N}} & \cdots & \left(e^{-\frac{j2\pi}{mN^2}}\right)^{\boldsymbol{[m(N-1)+2]N+N-1}} \\
\vdots & & \vdots \\
\left(e^{-\frac{j2\pi}{mN^2}}\right)^{\boldsymbol{(mN-1)N}} & \cdots & \left(e^{-\frac{j2\pi}{mN^2}}\right)^{\boldsymbol{(mN-1)N+N-1}}
\end{pmatrix}
$$

Linear phase shift

Proof - continued

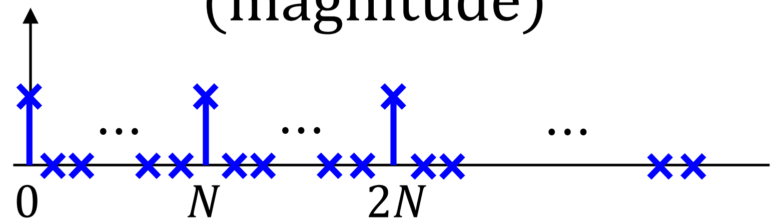


Proof - continued

Time domain

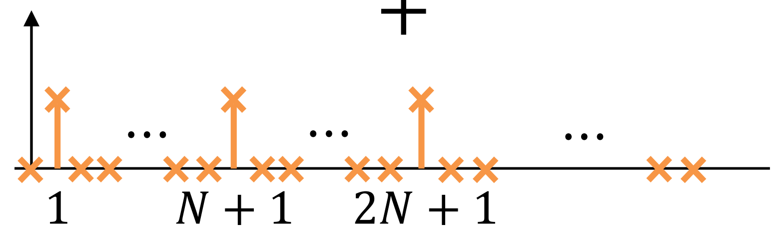
Frequency domain (magnitude)

$(\boldsymbol{s}_0, \mathbf{0}, \mathbf{0}, \dots, \mathbf{0})$ ⟷



$+$ $\quad$ $+$

$(\mathbf{0}, \boldsymbol{s}_1, \mathbf{0}, \mathbf{0}, \dots, \mathbf{0})$ ⟷

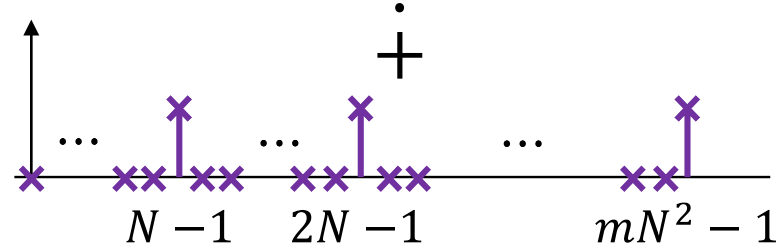


$+$ $\quad$ $+$

$\vdots$ $\quad$ $\vdots$

$+$ $\quad$ $+$

$(\mathbf{0}, \mathbf{0}, \dots, \mathbf{0}, \boldsymbol{s}_{N-1})$ ⟷



|| $\quad$ ||

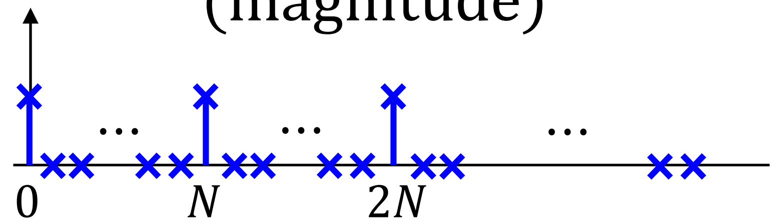
$(\boldsymbol{s}_0, \boldsymbol{s}_1, \boldsymbol{s}_2, \dots, \boldsymbol{s}_{N-1})$ ⟷ ?

Proof - done

Time domain

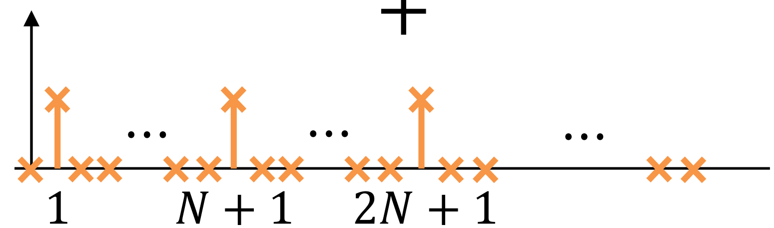
Frequency domain (magnitude)

$(s_0, 0, 0, \dots, 0)$



$+$

$(0, s_1, 0, 0, \dots, 0)$

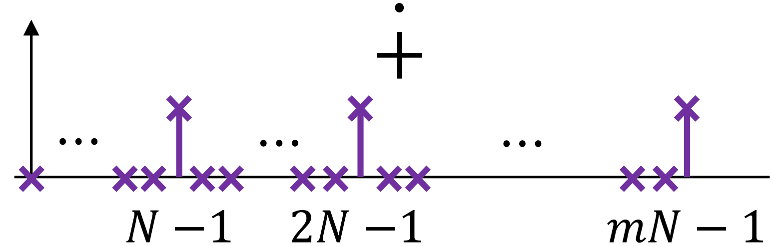


$+$

$\vdots$

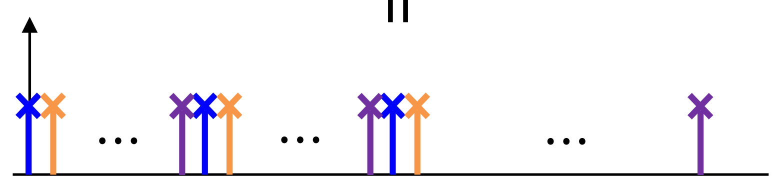
$+$

$(0, 0, \dots, 0, s_{N-1})$



||

$(s_0, s_1, s_2, \dots, s_{N-1})$



Flat

Examples

A perfect sequence

$\{0, -1, 1, 0, 1, 1\}$ → **Generalized Milewski Construction**

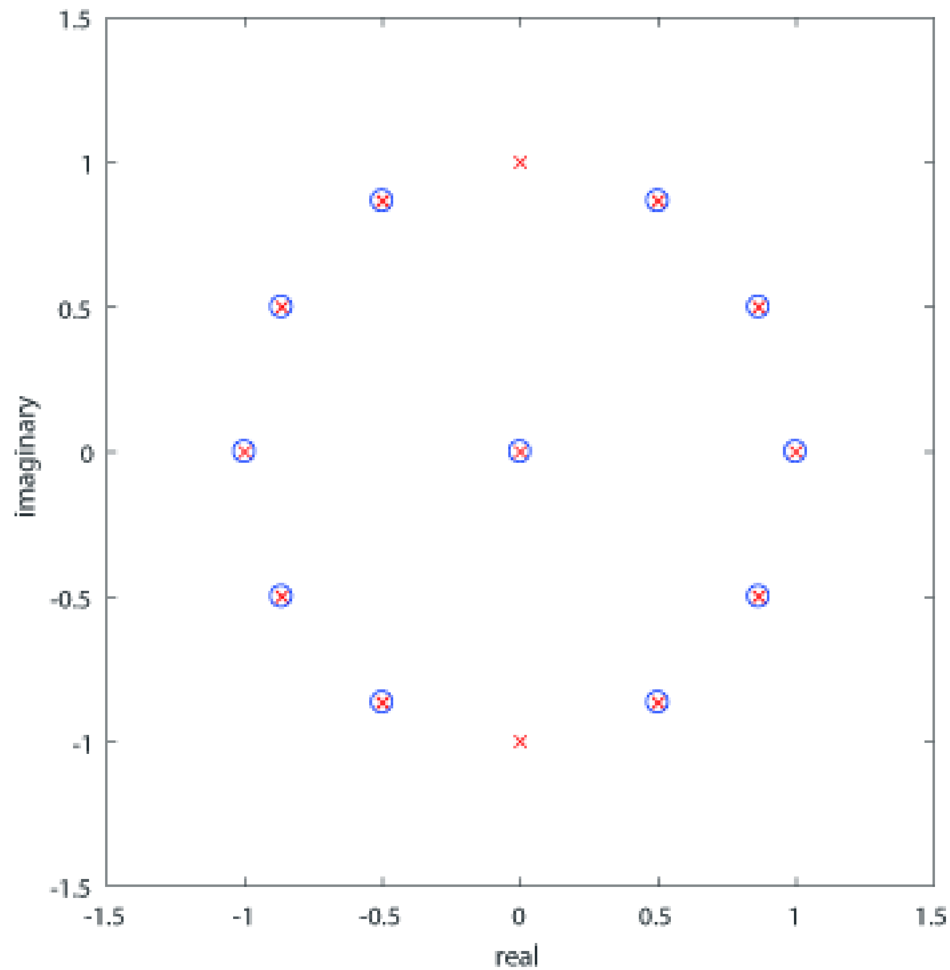
$N = 2$ → **Generalized Milewski Construction**

all-one → **Generalized Milewski Construction**

Generalized Milewski Construction →

$\boldsymbol{s} = \{0, 0, -1, -\omega, 1, \omega^2, 0, 0, 1, \omega^4, 1, \omega^5, 0, 0, -1, -\omega^7, 1, \omega^8, 0, 0, 1, \omega^{10}, 1, \omega^{11}\}$

※ $\omega = e^{-\frac{j2\pi}{12}}$



Constellation of $\boldsymbol{s}$: 12-PSK+

Examples (cont')

A perfect sequence

$\{3, -2, 3, -2, -2, 3, -2, -7, -2, -2\}$ of period 9

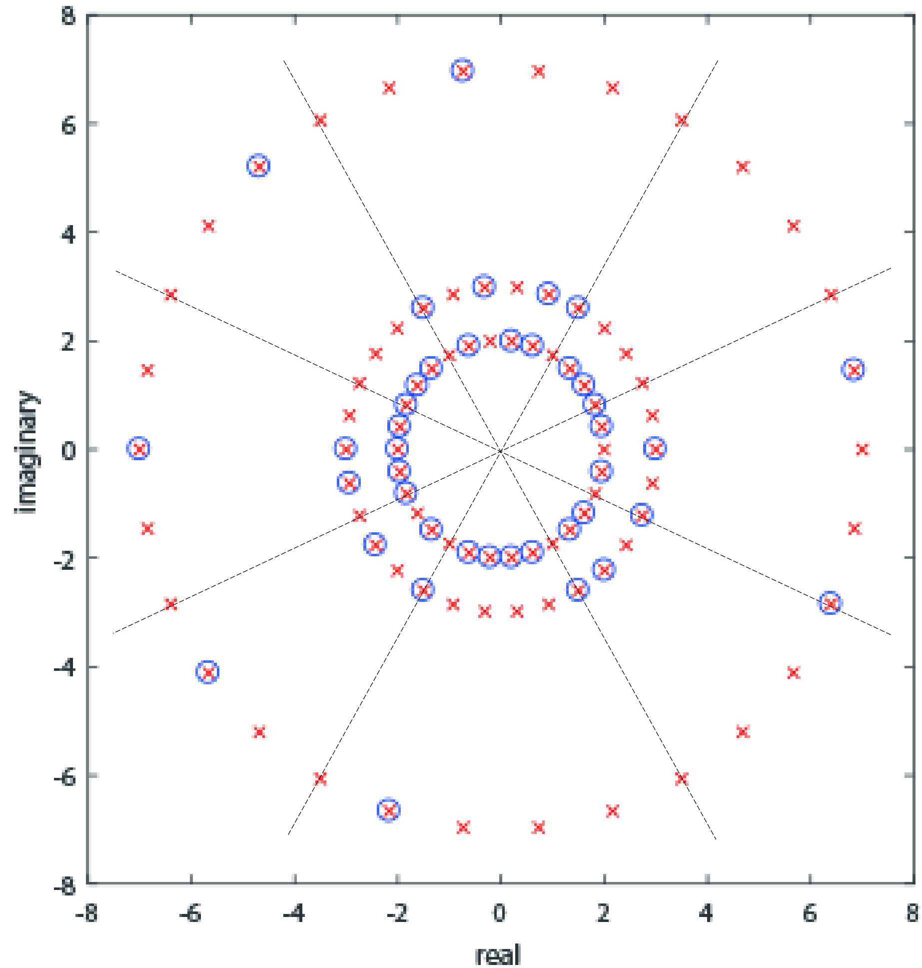
$N = 3$

all-one

Generalized Milewski Construction

$\boldsymbol{s}$ is a perfect sequence of period 90

Phases are aligned



Constellation of $\boldsymbol{s}$: APSK

(3 different amplitudes & 30 different phases)

How many different phases?

※ ω is an mN-th primitive root of unity

$\alpha(0)$	$\times$ $(\omega^0)^0$	$\alpha(0)$	$\times$ $(\omega^1)^0$	$\cdots$	$\alpha(0)$	$\times$ $(\omega^{N-1})^0$
$\alpha(1)$	$\times$ $(\omega^0)^1$	$\alpha(1)$	$\times$ $(\omega^1)^1$	$\cdots$	$\alpha(1)$	$\times$ $(\omega^{N-1})^1$
$\alpha(2)$	$\times$ $(\omega^0)^2$	$\alpha(2)$	$\times$ $(\omega^1)^2$	$\cdots$	$\alpha(2)$	$\times$ $(\omega^{N-1})^2$
	$\vdots$		$\vdots$	$\ddots$		$\vdots$
$\alpha(m-1)$	$\times$ $(\omega^0)^{m-1}$	$\alpha(m-1)$	$\times$ $(\omega^1)^{m-1}$	$\cdots$	$\alpha(m-1)$	$\times$ $(\omega^{N-1})^{m-1}$
	$\vdots$		$\vdots$	$\ddots$		$\vdots$
$\alpha(0)$	$\times$ $(\omega^0)^{m(N-1)}$	$\alpha(0)$	$\times$ $(\omega^1)^{m(N-1)}$	$\cdots$	$\alpha(0)$	$\times$ $(\omega^{N-1})^{m(N-1)}$
$\alpha(1)$	$\times$ $(\omega^0)^{m(N-1)+1}$	$\alpha(1)$	$\times$ $(\omega^1)^{m(N-1)+1}$	$\cdots$	$\alpha(1)$	$\times$ $(\omega^{N-1})^{m(N-1)+1}$
$\alpha(2)$	$\times$ $(\omega^0)^{m(N-1)+2}$	$\alpha(2)$	$\times$ $(\omega^1)^{m(N-1)+2}$	$\cdots$	$\alpha(2)$	$\times$ $(\omega^{N-1})^{m(N-1)+2}$
	$\vdots$		$\vdots$	$\ddots$		$\vdots$
$\alpha(m-1)$	$\times$ $(\omega^0)^{mN-1}$	$\alpha(m-1)$	$\times$ $(\omega^1)^{mN-1}$	$\cdots$	$\alpha(m-1)$	$\times$ $(\omega^{N-1})^{mN-1}$

mN different phases required, in general

For arbitrary chosen μ

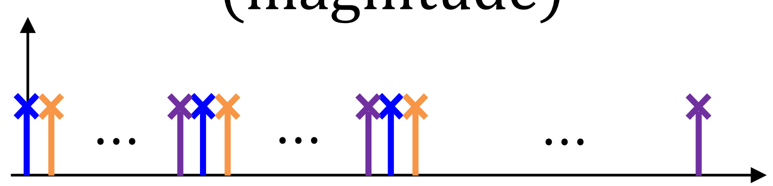
Multiplying a constant $\mu(r)$ with $|\mu(r)| = 1$
to the r-th column
does not affect on the magnitudes of the DFT of

$$(\mathbf{0}, \mathbf{0}, \dots, \mu(r)\boldsymbol{s}_r, \mathbf{0}, \dots, \mathbf{0}) = \mu(r)(\mathbf{0}, \mathbf{0}, \dots, \boldsymbol{s}_r, \mathbf{0}, \dots, \mathbf{0})$$

Time domain

$$(\mu(0)\boldsymbol{s}_0, \mu(1)\boldsymbol{s}_1, \dots, \mu(N-1)\boldsymbol{s}_{N-1})$$

Frequency domain
(magnitude)



Still flat !

A comparison

$\mathbf{gcd}(L_1, L_2) = \mathbf{1}$

perfect sequence $\boldsymbol{x} = \{x(n)\}_{n=0}^{L_1-1}$

perfect sequence $\boldsymbol{y} = \{y(n)\}_{n=0}^{L_2-1}$

Direct product construction

Output perfect sequence

$$\boldsymbol{z} = \{z(n)\}_{n=0}^{L_1L_2-1} = \{x(n)y(n)\}_{n=0}^{L_1L_2-1}$$

- Synthesize a long perfect sequence
- Need **two** short **perfect sequences** of **relatively prime period**
- The **constellation** depends on **input** perfect sequences

Perfect sequence $\boldsymbol{\alpha} = \{\alpha(n)\}_{n=0}^{m-1}$

A positive integer N

Any polyphase sequence of length N: $\boldsymbol{\mu}$

Generalized Milewski Construction

Output perfect sequence

$$\boldsymbol{s} = \{s(n)\}_{n=0}^{\boldsymbol{mN^2}-1} = \{\mu(r)\alpha(q)\omega^{qr}\}_{n=0}^{mN^2-1}$$

- Synthesize a long perfect sequence
- Need a **single** short **perfect sequence**
- **PSK+ or APSK+** constellations

Concluding remarks

- The Milewski construction is generalized in period and constellation
  - flexible period, various input sequences
  - perfect sequences over PSK, PSK+, APSK+
    - Example: Input perfect over ASK+ ➔ Output perfect over APSK+

- Some interesting questions for future works:
  - **Possible to reduce the number of phase?**
    - At least mN different phases, in general.
    - related to constructing perfect sequences over QAM constellation
  - **For practical APSK, how to tilt points?**
    - Two points on circles of different radius have different phase offset to maximize distance, in practice.