Milewski sequences revisited, and its generalization

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Sequences and Correlation

- For complex-valued sequences $\boldsymbol{x}, \boldsymbol{y}$ of length L, the periodic correlation of $\boldsymbol{x}$ and $\boldsymbol{y}$ at shift $\boldsymbol{\tau}$ is

$$C_{x,y}(\tau) = \sum_{n=0}^{L-1} x(n+\tau)y^*(n)$$

  - If $\boldsymbol{y}$ is a cyclic shift of $\boldsymbol{x}$, it is called **autocorrelation,** and denoted by $C_x(\tau)$
  - Otherwise, it is called **crosscorrelation**

Perfect Sequences

- A sequence $\boldsymbol{x}$ of length $\boldsymbol{L}$ is called **perfect** if

$$C_{\boldsymbol{x}}(\tau)=\begin{cases}\boldsymbol{E}, & \tau\equiv 0 \pmod{L}\\ 0, & \tau\not\equiv 0 \pmod{L}\end{cases}$$

Here, $\boldsymbol{E}$ is called the energy of $\boldsymbol{x}$

- (Sarwate, 79) Crosscorrelation of any two perfect sequences of length $\boldsymbol{L}$ with the same energy $\boldsymbol{E}$ is lower bounded by $\boldsymbol{E}/\sqrt{\boldsymbol{L}}$.
  - **An optimal pair** of perfect sequences of length $\boldsymbol{L}$
  - **An optimal set** of perfect sequences of length $\boldsymbol{L}$

Interleaved Sequence

- Consider two sequences $\boldsymbol{s}_0 = \{a, b, c\}$ and $\boldsymbol{s}_1 = \{d, e, f\}$ of length **3** each

- Write each as a column of an array:

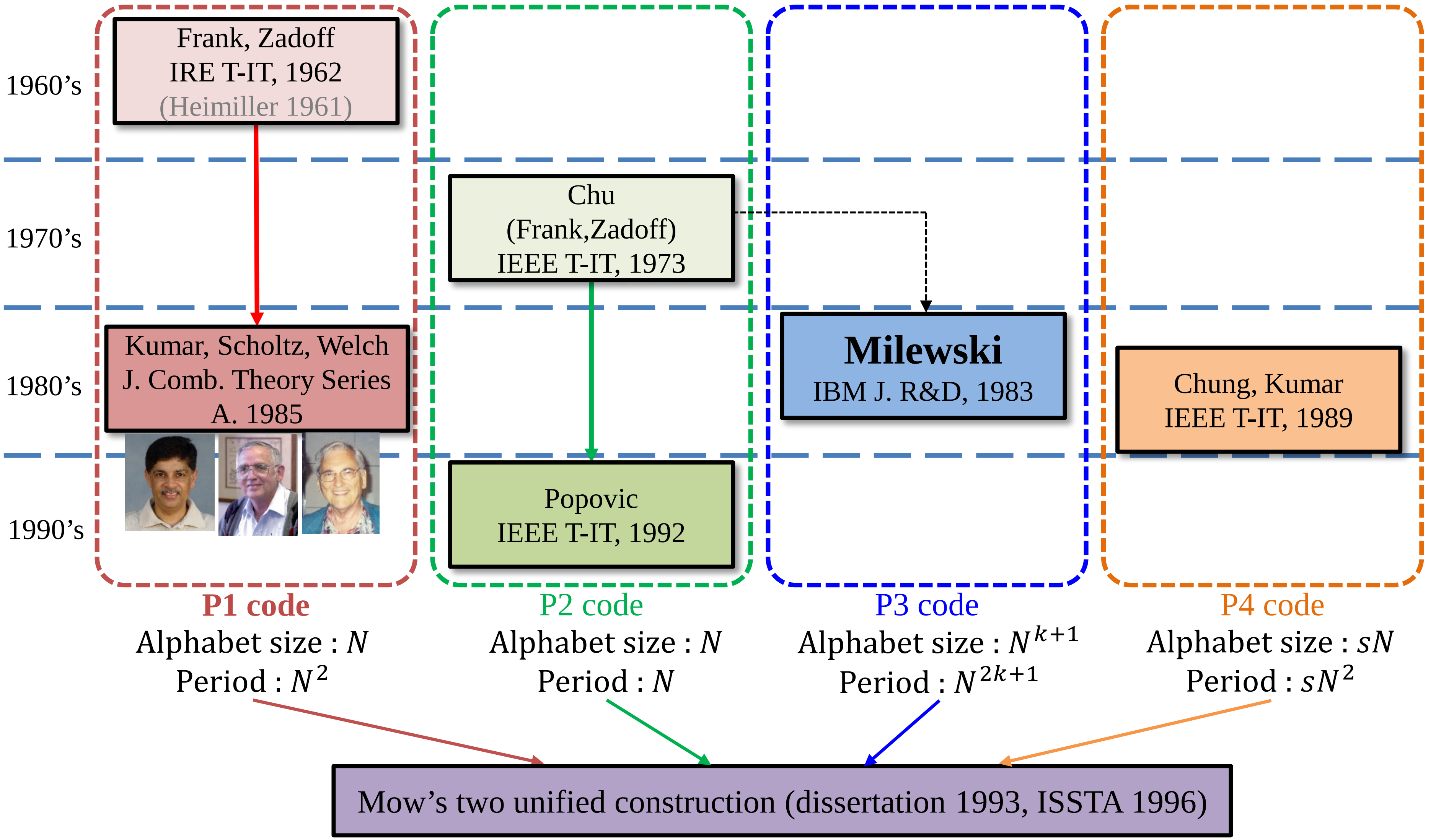
$$[\boldsymbol{s}_0, \boldsymbol{s}_1] = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$$

- Read the array row-by-row and obtain a sequence of length **6**:

$$\boldsymbol{s} = I(\boldsymbol{s}_0, \boldsymbol{s}_1) = \{a, d, b, e, c, f\}$$

is called **an interleaved sequence** of $\boldsymbol{s}_0$ and $\boldsymbol{s}_1$

History of Perfect Polyphase Sequences



The original Milewski construction

Length: $m \rightarrow m \cdot m^{2K}$

perfect polyphase Sequence of length m

$\boldsymbol{\beta} = \{\alpha(n)\}_{n=0}^{m-1}$

A positive integer

K

Milewski Construction

Output **perfect polyphase** sequence

$$\boldsymbol{s} = \{s(n)\}_{n=0}^{m^{2K+1}-1}$$

where

$$\boldsymbol{s(n) = \beta(q)\omega^{qr}}$$

$$\omega = e^{-j\frac{2\pi}{m^{1+K}}}$$

Here, we use

$$n = \boldsymbol{q}\boldsymbol{m^K} + \boldsymbol{r} \leftrightarrow (\boldsymbol{q}, \boldsymbol{r})$$

$m \cdot m^K \times m^K$ array form of $\boldsymbol{s}$

$$\begin{bmatrix}
\beta(0) \times \mathbf{1} & \beta(0) \times \mathbf{1} & \cdots & \beta(0) \times \mathbf{1} \\
\beta(1) \times \mathbf{1} & \beta(1) \times \boldsymbol{\omega} & \cdots & \beta(1) \times (\boldsymbol{\omega}^{N-1})^{1} \\
\beta(2) \times \mathbf{1} & \beta(2) \times \boldsymbol{\omega}^{2} & \cdots & \beta(2) \times (\boldsymbol{\omega}^{N-1})^{2} \\
\vdots & \vdots & \ddots & \vdots \\
\beta(m-1) \times \mathbf{1} & \beta(m-1) \times \boldsymbol{\omega}^{m-1} & \cdots & \beta(m-1) \times (\boldsymbol{\omega}^{N-1})^{m-1} \\
\vdots & \vdots & \ddots & \vdots \\
\beta(0) \times \mathbf{1} & \beta(0) \times \boldsymbol{\omega}^{m(N-1)} & \cdots & \beta(0) \times (\boldsymbol{\omega}^{N-1})^{m(N-1)} \\
\beta(1) \times \mathbf{1} & \beta(1) \times \boldsymbol{\omega}^{m(N-1)+1} & \cdots & \beta(1) \times (\boldsymbol{\omega}^{N-1})^{m(N-1)+1} \\
\beta(2) \times \mathbf{1} & \beta(2) \times \boldsymbol{\omega}^{m(N-1)+2} & \cdots & \beta(2) \times (\boldsymbol{\omega}^{N-1})^{m(N-1)+2} \\
\vdots & \vdots & \ddots & \vdots \\
\beta(m-1) \times \mathbf{1} & \beta(m-1) \times \boldsymbol{\omega}^{mN-1} & \cdots & \beta(m-1) \times (\boldsymbol{\omega}^{N-1})^{mN-1}
\end{bmatrix}$$

Input sequence of period m

$$\boldsymbol{N = m^K}$$

Our framework

(A special type of interleaved sequences)

not necessarily **polyphase**
not necessarily **all distinct**

A collection of N sequence of length m
$\boldsymbol{B} = \{\boldsymbol{\beta}_0, \boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_N\}$

A positive integer
N

A **polyphase** sequence of length N
$\boldsymbol{\mu}$

A function $\mathbb{Z}_N \longrightarrow \mathbb{Z}_{mN}$
π

↓

Interleaving technique

↓

Output sequence $s = \{\boldsymbol{s(n)}\}_{n=0}^{mN^2-1}$

where

$$\boldsymbol{s(n)} = \boldsymbol{\mu(r)\beta_r(q)\omega^{q\pi(r)}}$$

with $\boldsymbol{n} = \boldsymbol{qN} + \boldsymbol{r}$, and $\boldsymbol{\omega} = \mathbf{exp}(-\boldsymbol{j2\pi/mN})$.

Definition. We define $\boldsymbol{\mathcal{A}}(\boldsymbol{B}, \boldsymbol{\pi})$ be a family of interleaved sequences constructed by the above procedure using all possible polyphase sequences $\boldsymbol{\mu}$.

Array Form

Assume that $\boldsymbol{\mu}$ is the all-one sequence, ※ $\omega = e^{-j\frac{2\pi}{mN}}$

Column index $\boldsymbol{r} = \mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots, \boldsymbol{N} - \mathbf{1}$

Row index $\boldsymbol{q} = \mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots, \boldsymbol{mN} - \mathbf{1}$

$\beta_0(0) \times (\omega^{\pi(0)})^{0}$	$\beta_1(0) \times (\omega^{\pi(1)})^{0}$	$\cdots$	$\beta_{N-1}(0) \times (\omega^{\pi(N-1)})^{0}$
$\beta_0(1) \times (\omega^{\pi(0)})^{1}$	$\beta_1(1) \times (\omega^{\pi(1)})^{1}$	$\cdots$	$\beta_{N-1}(1) \times (\omega^{\pi(N-1)})^{1}$
$\beta_0(2) \times (\omega^{\pi(0)})^{2}$	$\beta_1(2) \times (\omega^{\pi(1)})^{2}$	$\cdots$	$\beta_{N-1}(2) \times (\omega^{\pi(N-1)})^{2}$
$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\beta_0(m-1) \times (\omega^{\pi(0)})^{(m-1)}$	$\beta_1(m-1) \times (\omega^{\pi(1)})^{m-1}$	$\cdots$	$\beta_{N-1}(m-1) \times (\omega^{\pi(N-1)})^{m-1}$
$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\beta_0(0) \times (\omega^{\pi(0)})^{m(N-1)}$	$\beta_1(0) \times (\omega^{\pi(1)})^{m(N-1)}$	$\cdots$	$\beta_{N-1}(0) \times (\omega^{\pi(N-1)})^{m(N-1)}$
$\beta_0(1) \times (\omega^{\pi(0)})^{m(N-1)+1}$	$\beta_1(1) \times (\omega^{\pi(1)})^{m(N-1)+1}$	$\cdots$	$\beta_{N-1}(1) \times (\omega^{\pi(N-1)})^{m(N-1)+1}$
$\beta_0(2) \times (\omega^{\pi(0)})^{m(N-1)+2}$	$\beta_1(2) \times (\omega^{\pi(1)})^{m(N-1)+2}$	$\cdots$	$\beta_{N-1}(2) \times (\omega^{\pi(N-1)})^{m(N-1)+2}$
$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\beta_0(m-1) \times (\omega^{\pi(0)})^{mN-1}$	$\beta_1(m-1) \times (\omega^{\pi(1)})^{mN-1}$	$\cdots$	$\beta_{N-1}(m-1) \times (\omega^{\pi(N-1)})^{mN-1}$

Input sequence $\boldsymbol{\beta}_0$ of period m

Input sequence $\boldsymbol{\beta}_1$ of period m

Input sequence $\boldsymbol{\beta}_{N-1}$ of period m

Input function $\pi: \mathbb{Z}_{\boldsymbol{N}} \longrightarrow \mathbb{Z}_{\boldsymbol{mN}}$

Milewski Construction is a Special Case

Milewski Construction

Inputs:
- perfect polyphase sequence $\boldsymbol{\beta} = \{\alpha(n)\}_{n=0}^{m-1}$
- A positive integer K

Output perfect polyphase sequence

$$\boldsymbol{s} = \{s(n)\}_{n=0}^{\boldsymbol{m^{2K+1}}-1}$$

where

$$s(n) = \beta(q)\omega^{q\boldsymbol{r}}$$

with $n = q\boldsymbol{m^K} + r$,

and

$$\omega = e^{-j\frac{2\pi}{\boldsymbol{m^{1+K}}}}.$$

=

Interleaving technique

Inputs:
- perfect polyphase sequences $\boldsymbol{\beta}_0 = \boldsymbol{\beta}_1 = \cdots = \boldsymbol{\beta}_{N-1}$: $B = \{\boldsymbol{\beta}_0, \boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_N\}$
- An integer $N = m^K$: N
- all-one sequence: $\boldsymbol{\mu}$
- The identity function $\pi(r) = r$: π

Output sequence

$$\boldsymbol{s} = \{s(n)\}_{n=0}^{\boldsymbol{m(m^K)^2}-1}$$

where

$$s(n) = \beta_{\boldsymbol{r}}(q)\omega^{q\boldsymbol{\pi(r)}} = \beta(q)\omega^{q\boldsymbol{r}}$$

with $n = q\boldsymbol{N} + r = q\boldsymbol{m^K} + r$, and

$$\omega = e^{-j\frac{2\pi}{m\boldsymbol{N}}} = e^{-j\frac{2\pi}{\boldsymbol{m^{1+K}}}}.$$

Condition on perfectness
(Main result 1)

Definition. Let $\boldsymbol{\pi}, \boldsymbol{\sigma}$ be two functions from $\mathbb{Z}_N$ to $\mathbb{Z}_{mN}$. We define

$$\boldsymbol{\Psi}_{\pi,\sigma}(\boldsymbol{\tau}) = \{\, \boldsymbol{x} \in \mathbb{Z}_N | \boldsymbol{\pi}(\boldsymbol{x}+\boldsymbol{\tau}) \equiv \boldsymbol{\sigma}(\boldsymbol{x}) \pmod{\boldsymbol{N}} \,\}.$$

When $\boldsymbol{\pi} = \boldsymbol{\sigma}$, we use $\boldsymbol{\Psi}_{\pi}(\boldsymbol{\tau})$ simply.

Theorem. Any sequence in $\mathcal{A}(\boldsymbol{B}, \boldsymbol{\pi})$ is perfect **if and only if** the following conditions are satisfied:

1) $|\boldsymbol{\Psi}_{\pi}(\boldsymbol{r})| = \mathbf{0}$ for $\boldsymbol{r} = \mathbf{1}, \mathbf{2}, \dots, \boldsymbol{N}-\mathbf{1}$.
   That is, $\boldsymbol{\pi}(\boldsymbol{r}) \pmod{\boldsymbol{N}}$ for $\boldsymbol{r} = \mathbf{0}, \mathbf{1}, \dots, \boldsymbol{N}-\mathbf{1}$ is a permutation over $\mathbb{Z}_N$.

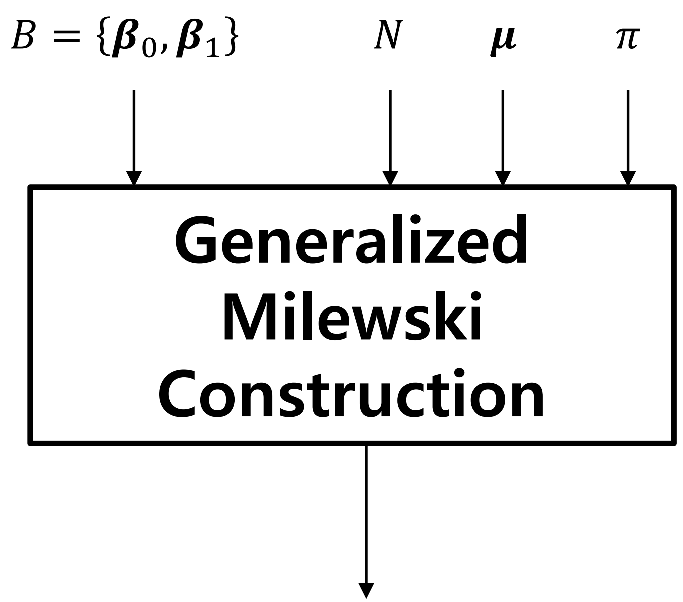
2) $\boldsymbol{B}$ is a collection of perfect sequences all of period m with the same energy.

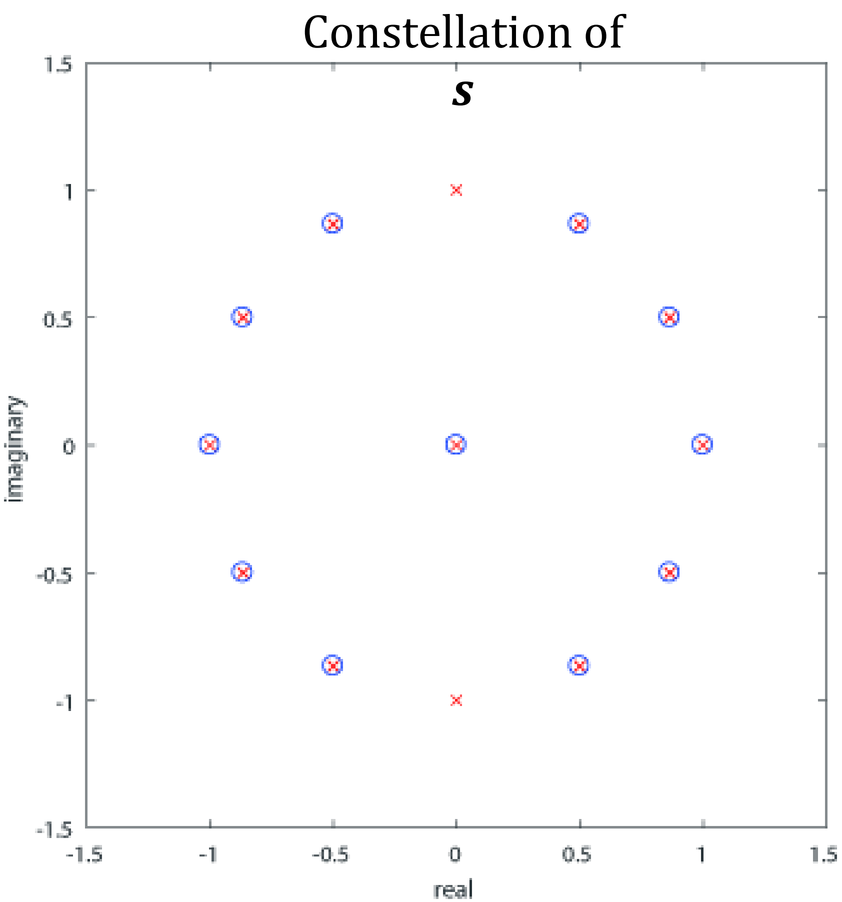
We now call them

the generalized Milewski sequences

Examples

- $\boldsymbol{\beta}_0 = \boldsymbol{\beta}_1 = \{0, -1, 1, 0, 1, 1\}$ which is a perfect sequence of length 6,
- $N = 2$,
- $\pi(r) = r$, and
- $\boldsymbol{\mu}$ is the all-one sequence.





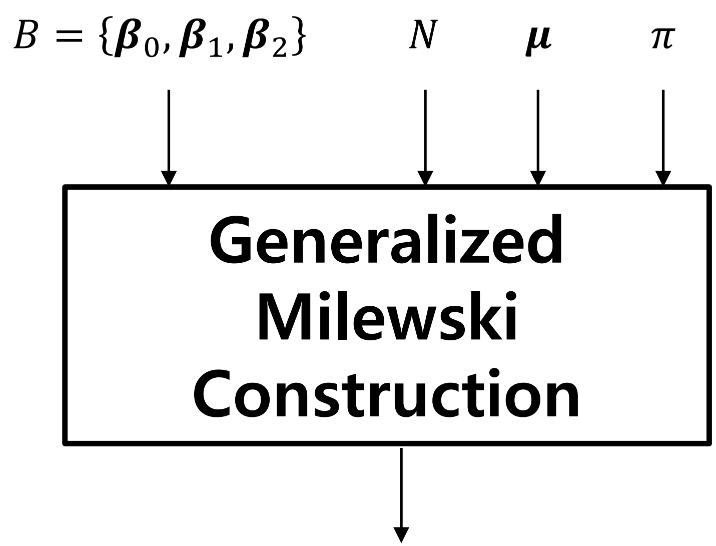
$\boldsymbol{s} = \{0, 0, -1, -\omega, 1, \omega^2, 0, 0, 1, \omega^4, 1, \omega^5, 0, 0, -1, -\omega^7, 1, \omega^8, 0, 0, 1, \omega^{10}, 1, \omega^{11}\}$ is a perfect sequence of length 24.

※ $\omega = e^{-j\frac{2\pi}{12}}$

Examples

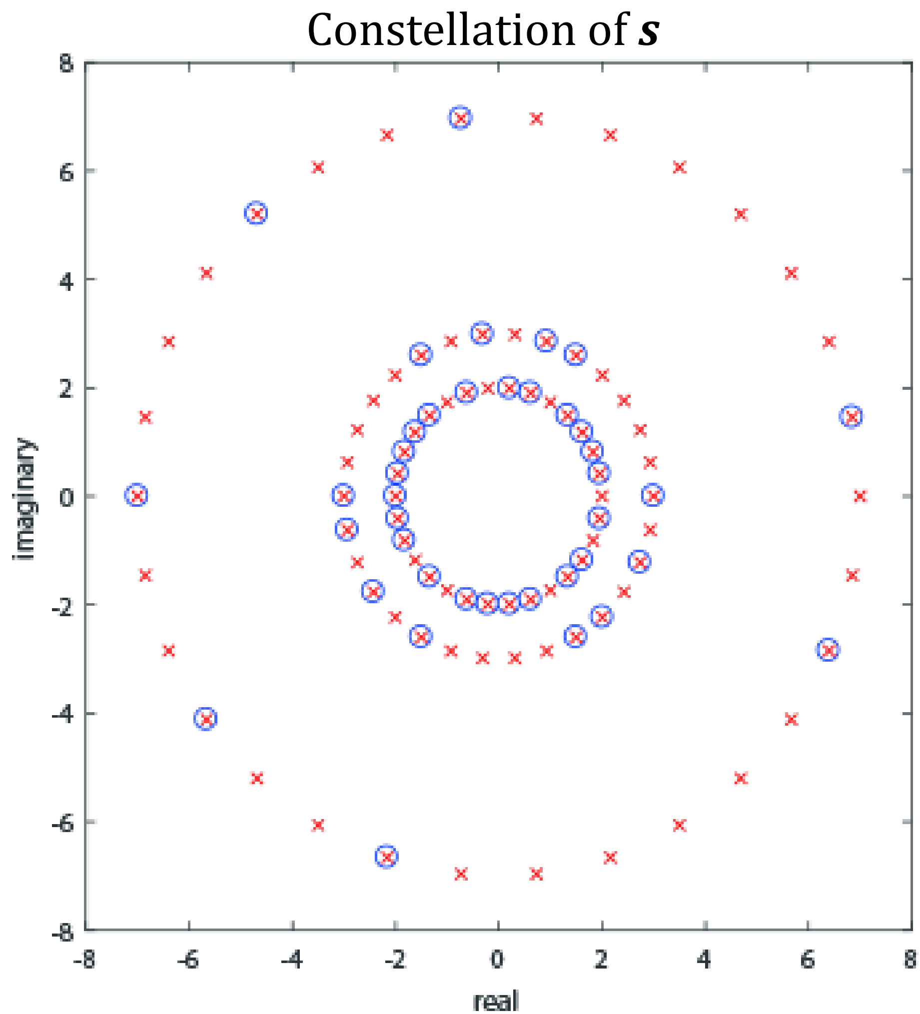
ASK constellation

- $\boldsymbol{\beta}_0 = \boldsymbol{\beta}_1 = \boldsymbol{\beta}_2 = \{3, -2, 3, -2, -2, 3, -2, -7, -2, -2\}$ which is a perfect sequence of period 10
- $N = 3$,
- $\pi(r) = r$, and
- $\boldsymbol{\mu}$ is the all-one sequence.



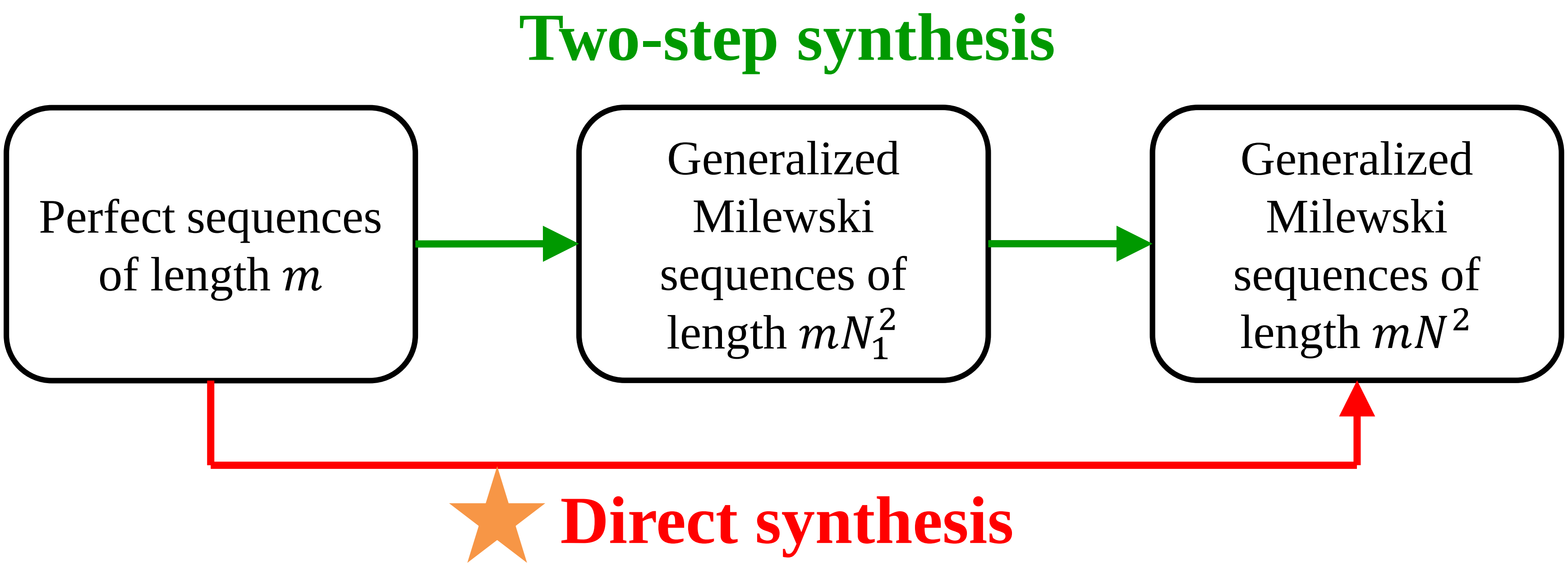
$\boldsymbol{s}$ is a perfect sequence of length 90.

※ $\omega = e^{-j\frac{2\pi}{12}}$



APSK constellation

Direct vs Indirect



Theorem. Assume that N is a composite number.

1) Any generalized Milewski sequence of length mN^2 from the two-step method can be also obtained by the direct method.
2) There exists a generalized Milewski sequence of length mN^2 from the direct method which can not be obtained by the two-step method.

Condition on optimal pair
(Main result 2)

Theorem. Let $B_1 = \{\boldsymbol{\beta}_0, \boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_{N-1}\}$ and $B_2 = \{\boldsymbol{\gamma}_0, \boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_{N-1}\}$, all of length m and the same energy E_B, and perfect.
Construct $\boldsymbol{s} \in \boldsymbol{\mathcal{A}}(B_1, \pi)$ and $\boldsymbol{f} \in \boldsymbol{\mathcal{A}}(B_2, \sigma)$.

Then, $\boldsymbol{s}$ and $\boldsymbol{f}$ have **optimal crosscorrelation** **if and only if** the following conditions are satisfied for each $\boldsymbol{r} = \boldsymbol{0}, \boldsymbol{1}, \dots, \boldsymbol{N} - \boldsymbol{1}$:

1) $|\Psi_{\pi,\sigma}(r)| = 1$, i.e., $\Psi_{\pi,\sigma}(r) = \{x\}$.
2) For the unique $x \in \Psi_{\pi,\sigma}(r)$, the pair of sequences

$$\left\{\beta_{x+r}(t)\omega_m^{\pi(x+r)t}\right\}_{t=0}^{m-1} \quad \text{and} \quad \left\{\gamma_x(t)\omega_m^{\sigma(x)t}\right\}_{t=0}^{m-1}$$ is optimal.

Condition on optimal pair
(Simple Special Case)

Corollary. Let $B_1 = \{\boldsymbol{\beta}_0, \boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_{N-1}\}$ and $B_2 = \{\boldsymbol{\gamma}_0, \boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_{N-1}\}$, all of length m and the same energy E_B, and perfect.

Assume that $\boldsymbol{\pi}$ and $\boldsymbol{\sigma}$ have the same range.

Construct $\boldsymbol{s} \in \boldsymbol{\mathcal{A}}(B_1, \pi)$ and $\boldsymbol{f} \in \boldsymbol{\mathcal{A}}(B_2, \sigma)$.

Then, $\boldsymbol{s}$ and $\boldsymbol{f}$ have **optimal crosscorrelation** **if and only if** the following conditions are satisfied for each $\boldsymbol{r = 0, 1, \dots, N - 1}$:

1) $|\Psi_{\pi,\sigma}(r)| = 1$, i.e., $\Psi_{\pi,\sigma}(r) = \{x\}$.
2) For the unique $x \in \Psi_{\pi,\sigma}(r)$,

the pair of sequences $\boldsymbol{\beta}_{x+r}$ and $\boldsymbol{\gamma}_x$ is optimal.

when $m = 1$

- The all-one sequence of length 1 is a trivial perfect sequence.
- And, we can say that

“the all-one sequence and itself is a (trivial) optimal pair of perfect sequences of length 1”

- Therefore, for $m = 1$,

an optimal k-set of **generalized Milewski sequences of length N^2** exists
if and only if
a $k \times N$ **circular Florentine array** exists

Example

- For a 4×15 circular Florentine array

Song 00

π_1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
π_2	0	7	1	8	2	12	3	11	9	4	13	5	14	6	10
π_3	0	4	11	7	10	1	13	9	5	8	3	6	2	14	12
π_4	0	13	7	2	11	6	14	10	3	5	12	9	1	4	8

we have **optimal 4-set of generalized Milewski sequences** of length $N^2 = 15^2$ by picking up a single perfect sequence from each and every

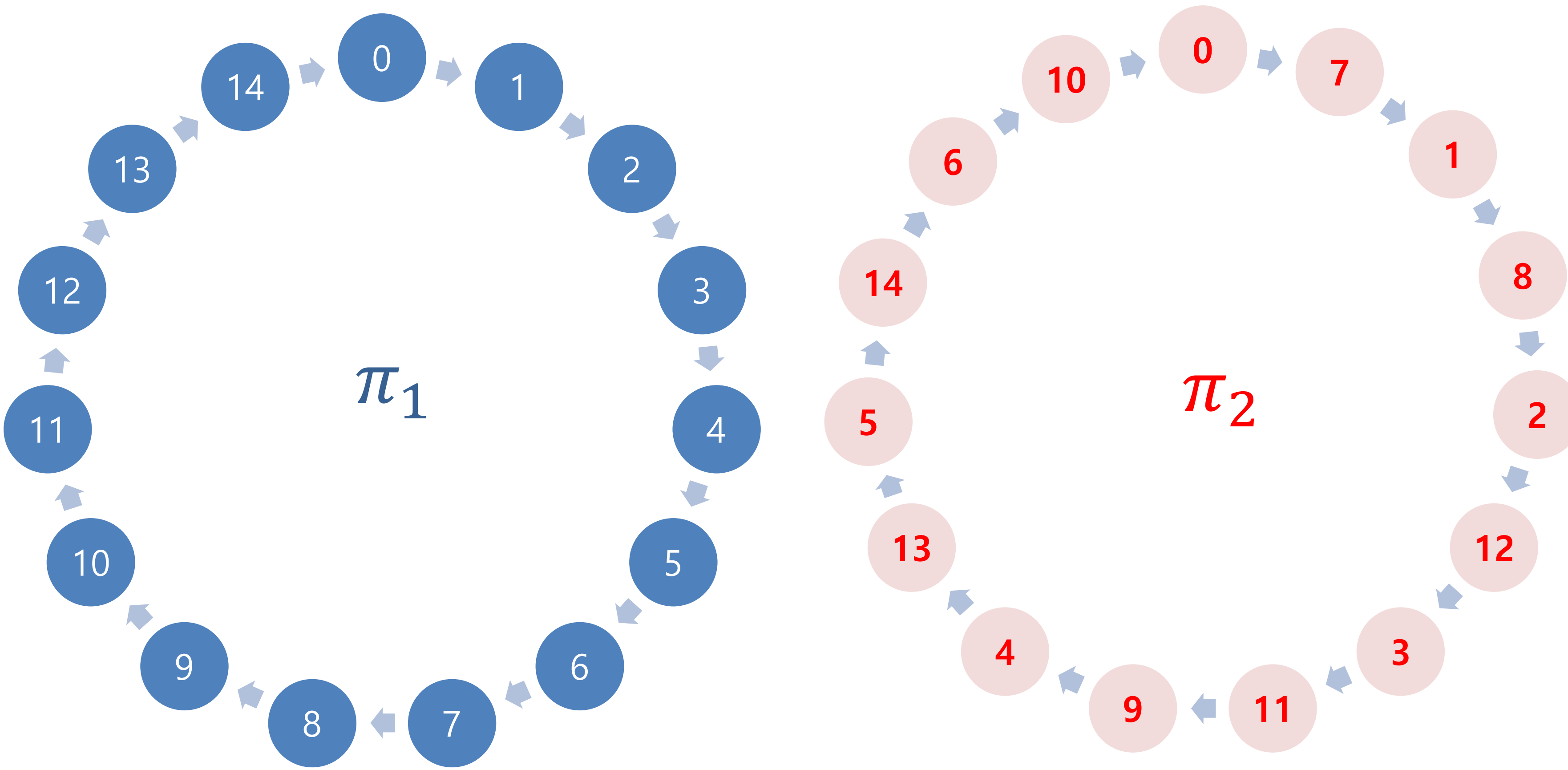
$$\mathcal{A}(\{1\}, \pi_1),\ \mathcal{A}(\{1\}, \pi_2),\ \mathcal{A}(\{1\}, \pi_3),\ \text{and}$$

$$\mathcal{A}(\{1\}, \pi_4).$$

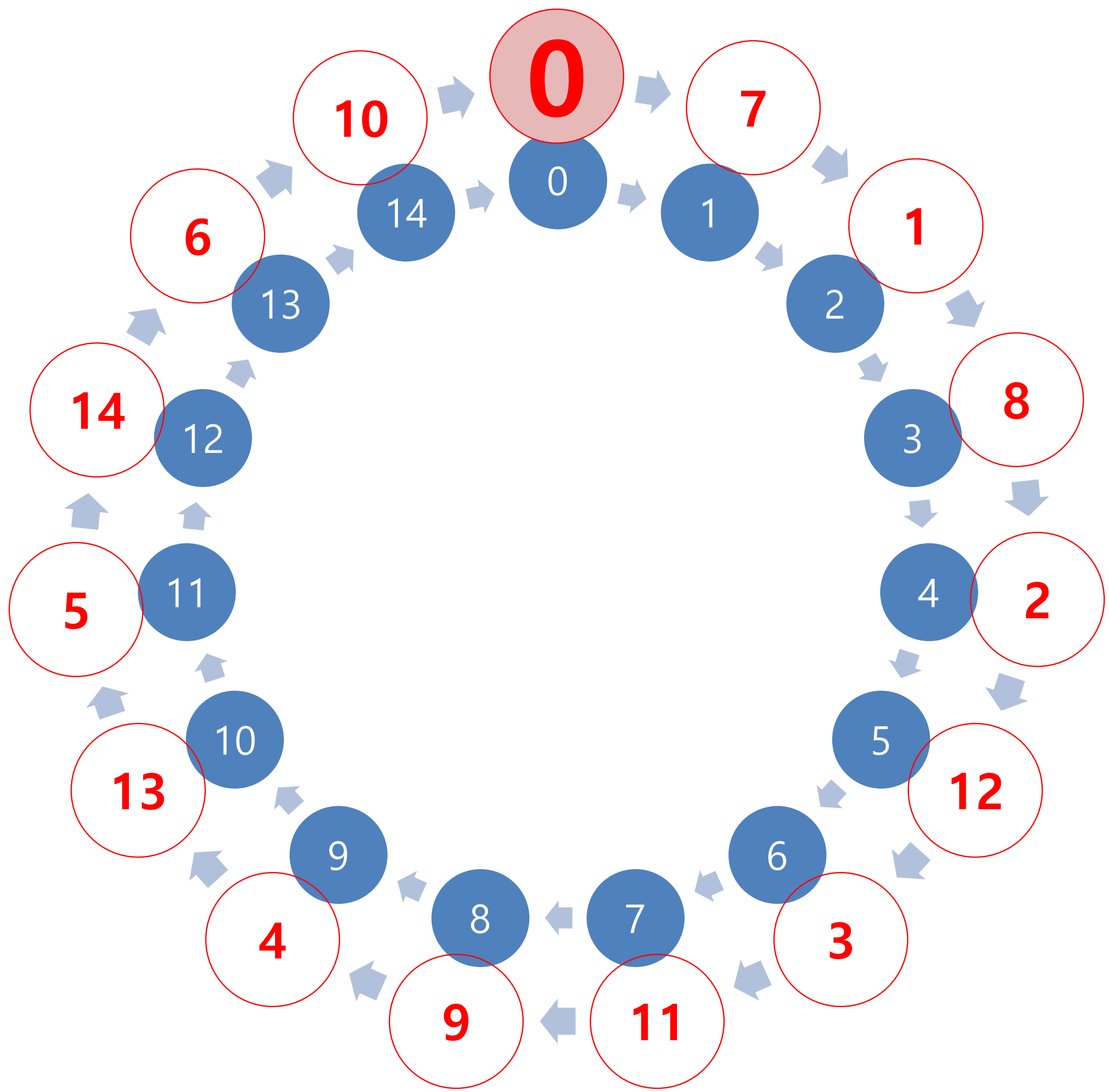
Check

$$\pi_2(x+\tau) = \pi_1(x)$$

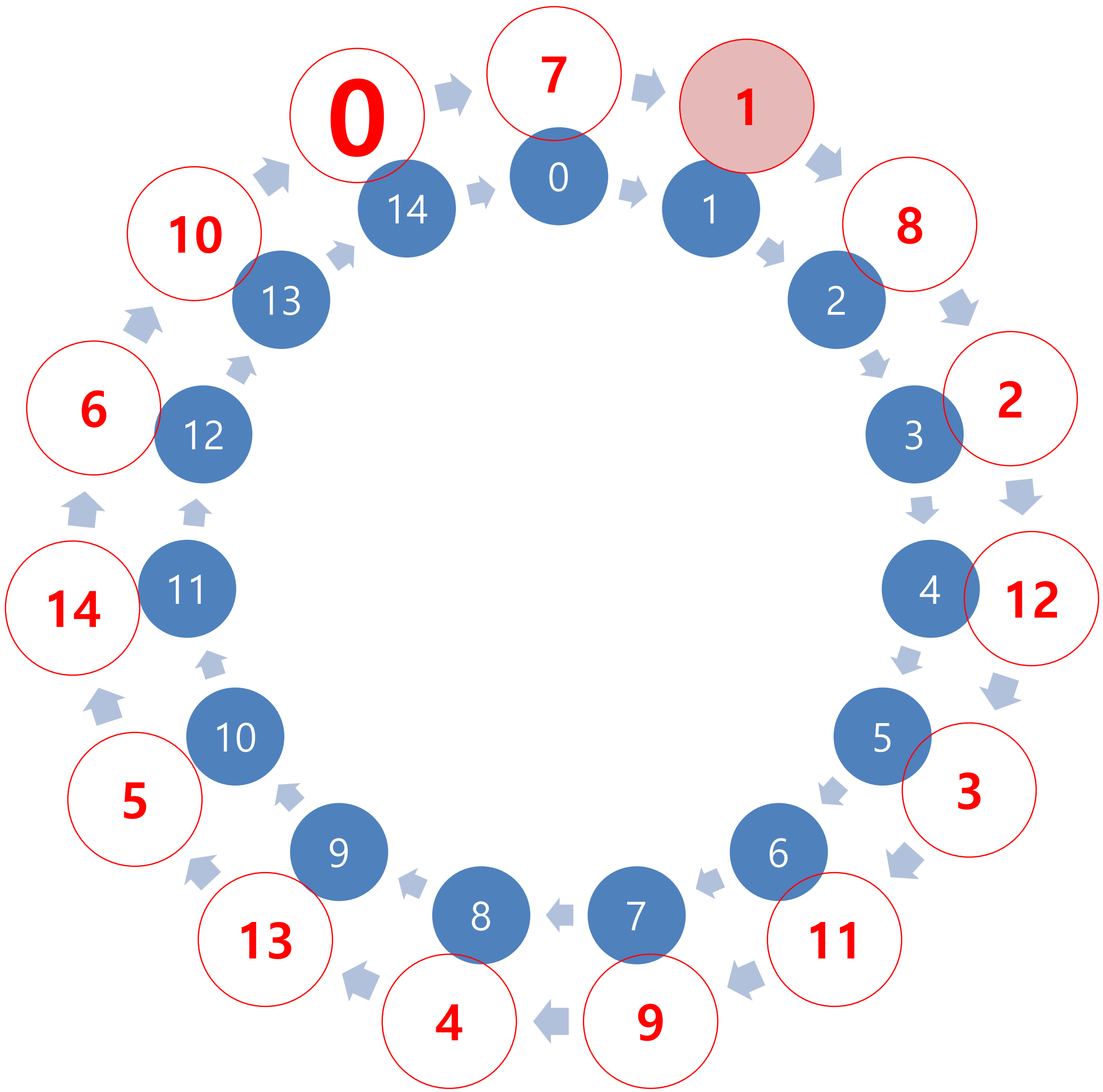
has exactly one solution x for any τ



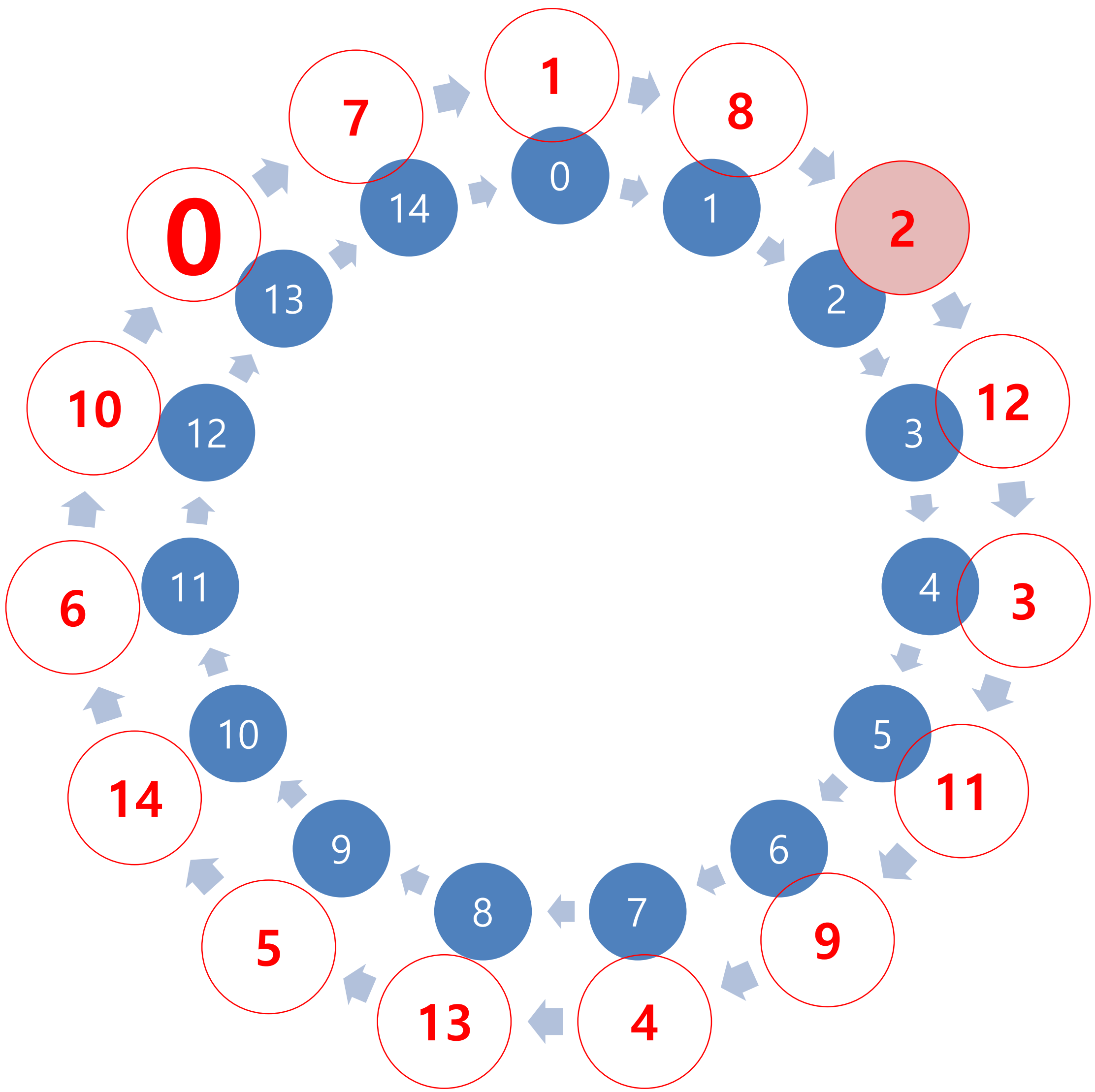
$$\tau = 0$$



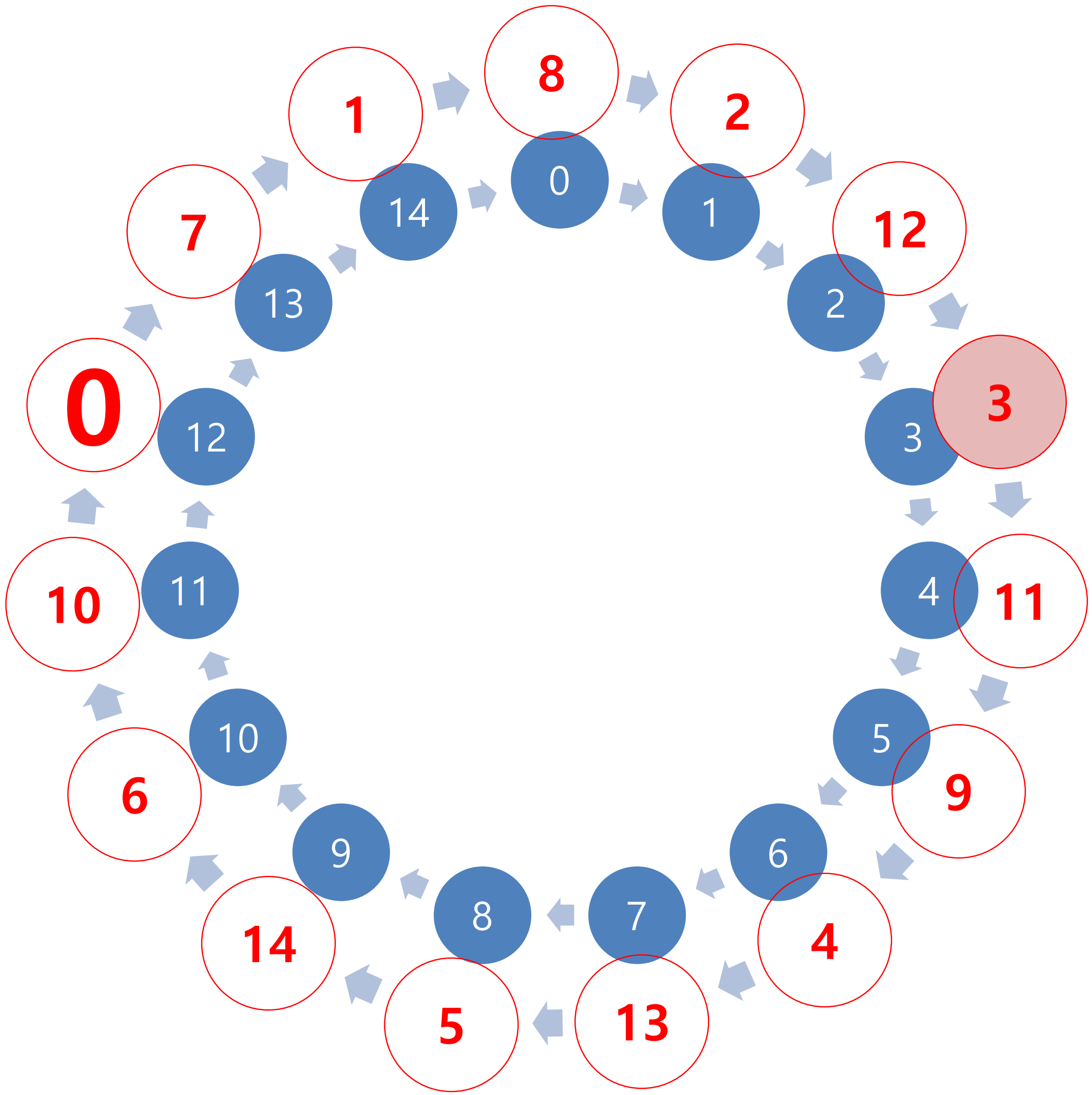
$$\tau = 1$$



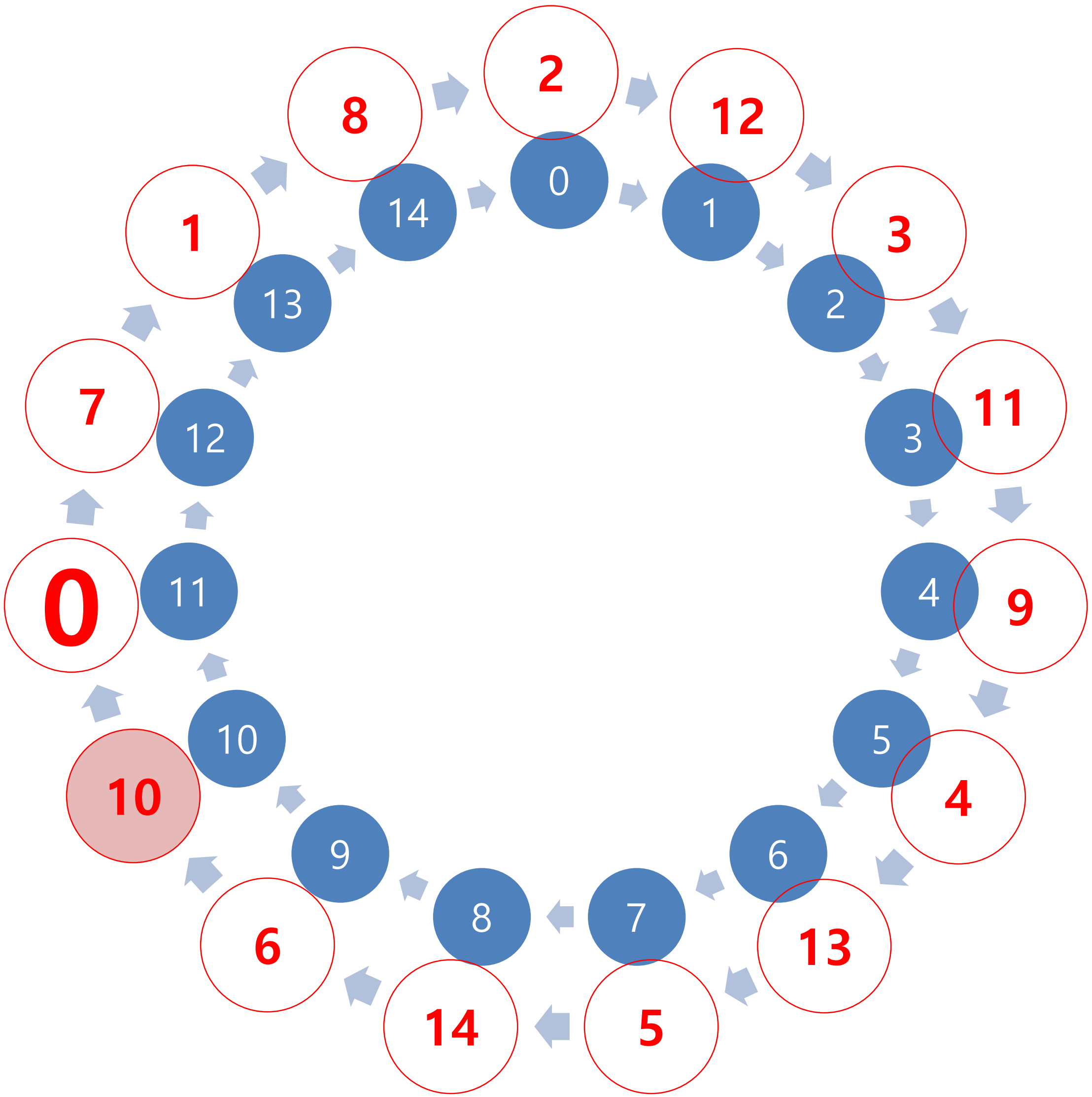
$$\tau = 2$$



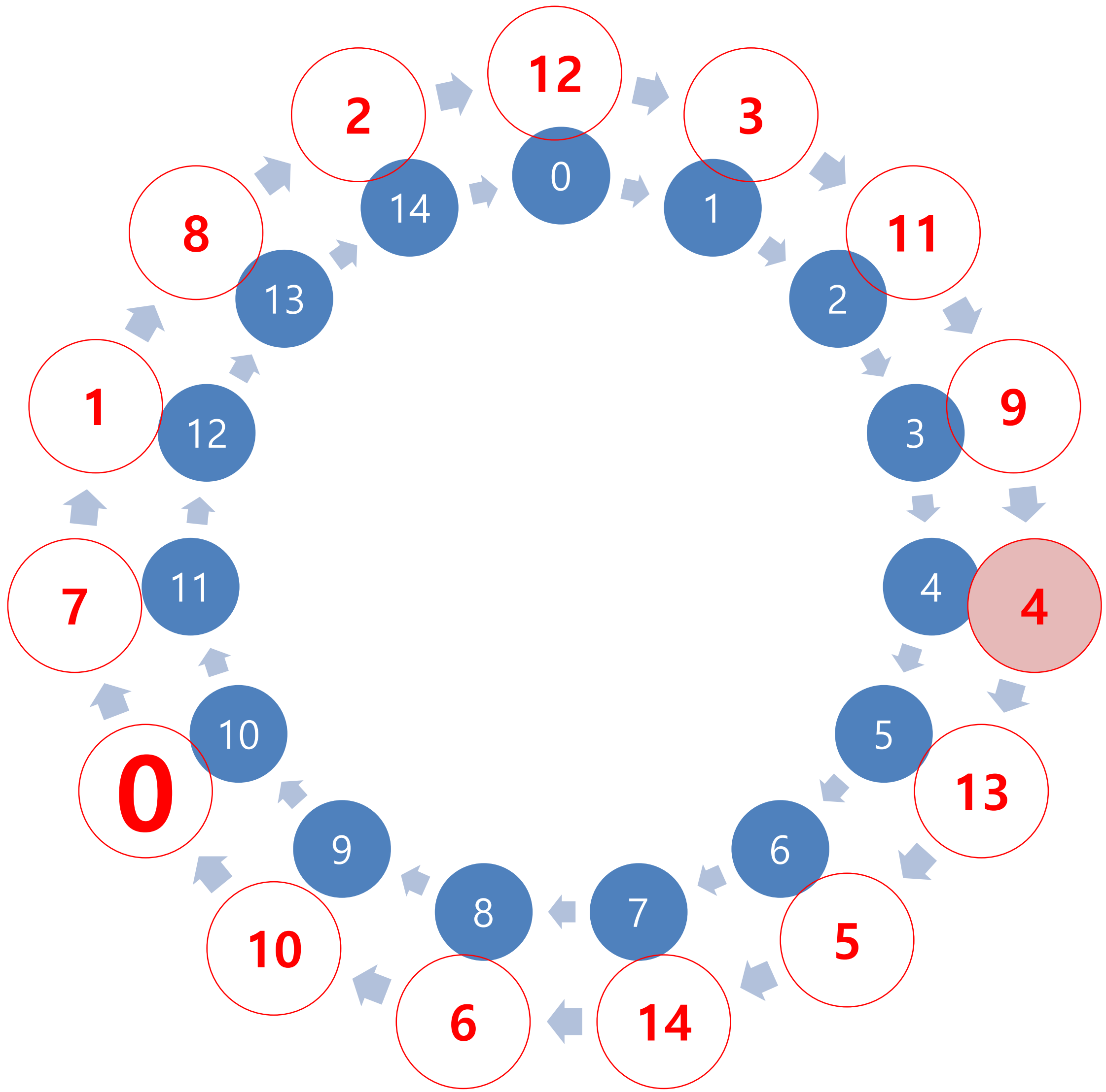
$\tau = 3$



$\tau = 4$

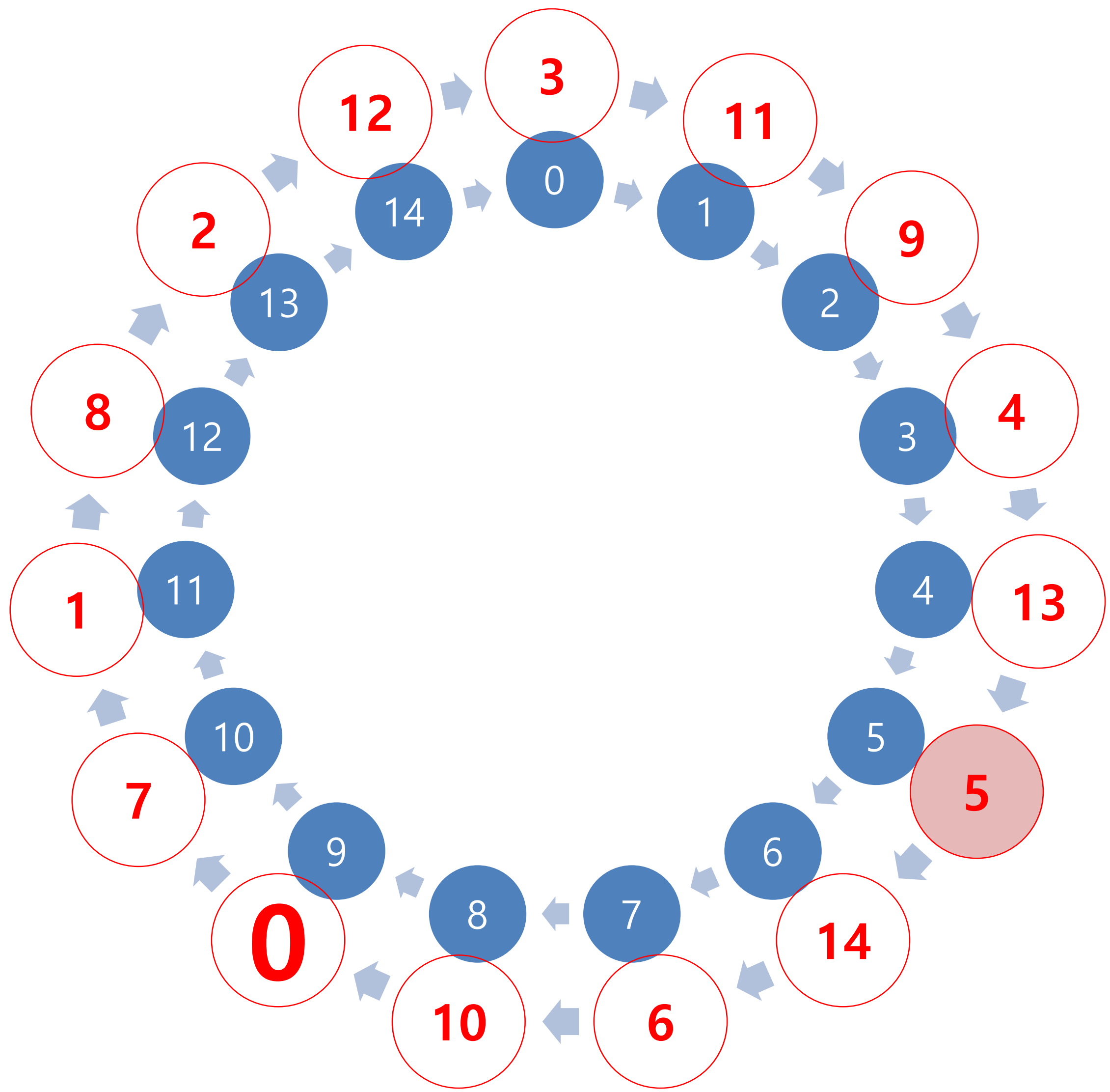


$\tau = 5$



$\tau = 6$

etc...



when $m > 1$

Assume we have an optimal pair $\boldsymbol{\beta}, \boldsymbol{\gamma}$ and **a 2×5 circular Florentine array:**

π_1	0	1	2	3	4
π_2	0	2	4	1	3

1) $\left|\Psi_{\pi,\sigma}(r)\right| = 1$, i.e., $\Psi_{\pi,\sigma}(r) = \{x\}$.
2) For the unique $x \in \Psi_{\pi,\sigma}(r)$, the pair of sequences $\boldsymbol{\beta}_{x+r}$ and $\boldsymbol{\gamma}_x$ is optimal.

- Construct $s \in \mathcal{A}(B_1, \pi_1)$, $f \in \mathcal{A}(B_2, \pi_2)$ with $B_1 = \{\beta_0, \beta_1, \dots, \beta_{N-1}\}$ and $B_2 = \{\gamma_0, \gamma_1, \dots, \gamma_{N-1}\}$, where

$$\boldsymbol{\beta}_0 = \boldsymbol{\gamma} \qquad \boldsymbol{\gamma}_0 = \boldsymbol{\beta}$$
$$\boldsymbol{\beta}_1 = \boldsymbol{\beta} \qquad \boldsymbol{\gamma}_1 = \boldsymbol{\gamma}$$
$$\vdots \qquad \vdots$$
$$\boldsymbol{\beta}_4 = \boldsymbol{\beta} \qquad \boldsymbol{\gamma}_4 = \boldsymbol{\gamma}$$

Then, **any** $s \in \mathcal{A}(B_1, \pi_1)$ and $f \in \mathcal{A}(B_2, \pi_2)$ is an **optimal pair**

Definition. Let $\boldsymbol{\pi}, \boldsymbol{\sigma}$ be two functions from $\mathbb{Z}_N$ to $\mathbb{Z}_{mN}$.

$$\boldsymbol{\Psi_{\pi,\sigma}(\tau) = \{\, x \in \mathbb{Z}_N \,|\, \pi(x+\tau) \equiv \sigma(x) \pmod{N} \,\}}.$$

π_1	0	1	2	3	4
π_2	0	2	4	1	3

$$\boldsymbol{\Psi_{1,2}(r) = \{\, x \in \mathbb{Z}_N \,|\, \pi_1(x+r) \equiv \pi_2(x) \pmod{5} \,\} \leftrightarrow \beta_{x+r} \text{ and } \gamma_x}$$

$\Psi_{1,2}(\mathbf{0}) = \{\, x \in \mathbb{Z}_N | \pi_1(x+\mathbf{0}) \equiv \pi_2(x) \pmod{5} \,\} = \{0\} \leftrightarrow \beta_{0+\mathbf{0}} = \beta_{\mathbf{0}}$ and $\gamma_{\mathbf{0}}$

$\Psi_{1,2}(\mathbf{1}) = \{\, x \in \mathbb{Z}_N | \pi_1(x+\mathbf{1}) \equiv \pi_2(x) \pmod{5} \,\} = \{2\} \leftrightarrow \beta_{2+\mathbf{1}} = \beta_{\mathbf{3}}$ and $\gamma_{\mathbf{2}}$

$\Psi_{1,2}(\mathbf{2}) = \{\, x \in \mathbb{Z}_N | \pi_1(x+\mathbf{2}) \equiv \pi_2(x) \pmod{5} \,\} = \{4\} \leftrightarrow \beta_{4+\mathbf{2}} = \beta_{\mathbf{1}}$ and $\gamma_{\mathbf{4}}$

$\Psi_{1,2}(\mathbf{3}) = \{\, x \in \mathbb{Z}_N | \pi_1(x+\mathbf{3}) \equiv \pi_2(x) \pmod{5} \,\} = \{1\} \leftrightarrow \beta_{1+\mathbf{3}} = \beta_{\mathbf{4}}$ and $\gamma_{\mathbf{1}}$

$\Psi_{1,2}(\mathbf{4}) = \{\, x \in \mathbb{Z}_N | \pi_1(x+\mathbf{4}) \equiv \pi_2(x) \pmod{5} \,\} = \{3\} \leftrightarrow \beta_{3+\mathbf{4}} = \beta_{\mathbf{2}}$ and $\gamma_{\mathbf{3}}$

$(\boldsymbol{\beta}_0\ \boldsymbol{\gamma}_0) = (\boldsymbol{\beta}, \boldsymbol{\gamma})$ **or** $(\boldsymbol{\gamma}, \boldsymbol{\beta})$

$(\boldsymbol{\beta}_1\ \boldsymbol{\gamma}_4) = (\boldsymbol{\beta}, \boldsymbol{\gamma})$ **or** $(\boldsymbol{\gamma}, \boldsymbol{\beta})$

$(\boldsymbol{\beta}_2\ \boldsymbol{\gamma}_3) = (\boldsymbol{\beta}, \boldsymbol{\gamma})$ **or** $(\boldsymbol{\gamma}, \boldsymbol{\beta})$

$(\boldsymbol{\beta}_3\ \boldsymbol{\gamma}_2) = (\boldsymbol{\beta}, \boldsymbol{\gamma})$ **or** $(\boldsymbol{\gamma}, \boldsymbol{\beta})$

$(\boldsymbol{\beta}_4\ \boldsymbol{\gamma}_1) = (\boldsymbol{\beta}, \boldsymbol{\gamma})$ **or** $(\boldsymbol{\gamma}, \boldsymbol{\beta})$

Maximum set size

Theorem. Let $F_c(N)$ be the maximal size of circular Florentine arrays with N columns(symbols). Denote by $O_G(mN^2)$ the maximum size of optimal sets generalized Milewski sequences of length mN^2 from perfect sequences of length m.

1) Assume that $m = 1$. Then

$$O_G(\mathrm{mN}^2) = F_c(N).$$

2) Assume that $m \geq 2$ and let $O_P(m)$ be the maximum size of optimal perfect sequence sets of period m. Then,

$$O_G = \min\{O_P(m), F_c(N)\}.$$

Maximum set size – polyphase

(Popovic, 1992)

The maximum size of optimal Zadoff-Chu sequence sets with period m is $p_{\min} - 1$, where $p_{\min}$ is the smallest prime factor of m.

Corollary. Let $N = mN^2$ be odd and let $O_M(L)$ be the maximum size of optimal sets of generalized Milewski polyphase sequences of length L constructed by using Zadoff-Chu sequences of length m.

1) If $m = 1$, then

$$O_M(L) = F_c(N).$$

2) If $m \geq 2$, then

$$O_M(L) = \min\{p_{\min} - 1, F_c(N)\},$$

where $p_{\min} - 1$ is the smallest prime factor of m.

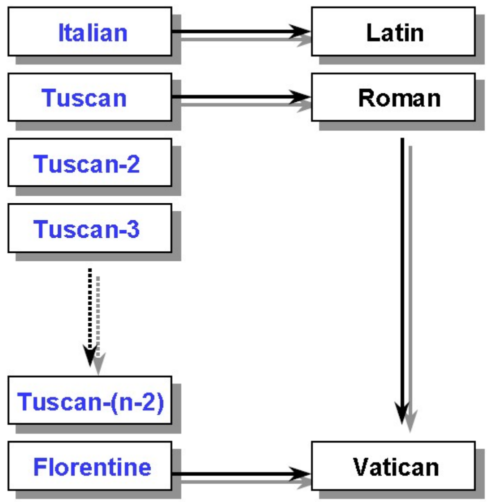
There is no optimal pair of generalized Milewski polyphase sequences of even length constructed by using Zadoff-chu sequences.

Concluding remarks

- To obtain an **optimal k-set of generalized Milewski sequences of length mN^2**, we need both:
  - **A $k \times N$ circular Florentine array**, and
  - **An optimal k-set of perfect sequences of length m**.

10 x 11 circular Florentine array

0	1	2	3	4	5	6	7	8	9	a
0	2	4	6	8	a	1	3	5	7	9
0	3	6	9	1	4	7	a	2	5	8
0	4	8	1	5	9	2	6	a	3	7
0	5	a	4	9	3	8	2	7	1	6
0	6	1	7	2	8	3	9	4	a	5
0	7	3	a	6	2	9	5	1	8	4
0	8	5	2	a	7	4	1	9	6	3
0	9	7	5	3	1	a	8	6	4	2
0	a	9	8	7	6	5	4	3	2	1



T. Etzion, S. W. Golomb and H. Taylor,
"Tuscan-K squares,"
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Concluding remarks

- To obtain an **optimal k-set of generalized Milewski sequences of length mN^2**, we need both:
  - **A $k \times N$ circular Florentine array**, and
  - **An optimal k-set of perfect sequences of length m**.

Some open problems:

- For a given integer N, what is the exact value of $F_c(N)$?
- For a given integer and its smallest prime factor $p_{\min}$, is there any other optimal set of size greater then $p_{\min} - 1$?

Thanks !