Some notes on the binary sequences of length $2^n - 1$ with the run property

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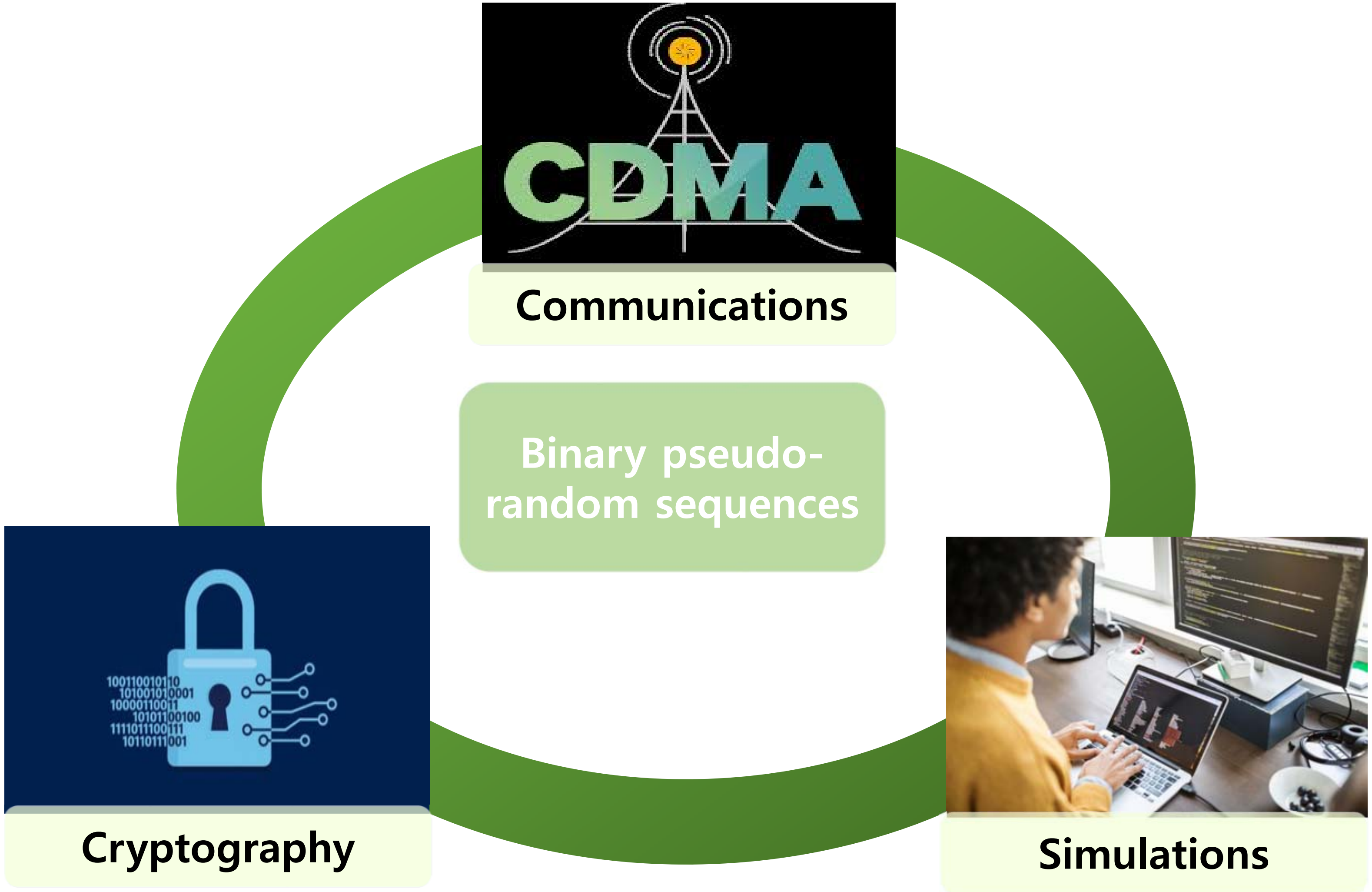
I. Introduction

PN sequence application



Randomness characteristics

Randomness (binary sequences)

Ideal autocorrelation

Many studies are conducted in terms of cyclic hadamard matrix [3], [11].

Multiplier

Helpful in the studies on ideal autocorrelation as a necessary condition [1], [7].

Span

-Correspondence with the de-Bruijn sequences

- Many studies are conducted in terms of their construction and some property [8], [9].

Run

Has not much related research results.

In this talk, ...

In this talk,

- calculate the number of run sequences of length $2^n - 1$

- present some interesting properties of the run sequences of length $2^n - 1$.

II. The number of run sequences of length $2^n - 1$

Run sequence

Definition 1. Run property [4], [5], [10]

A binary sequence of length $2^n - 1$ is said to have the **run property** if it has the run distribution as shown in the following table. For simplicity, we call such a sequence **a run sequence.**

Length	# of 1's run	# of 0's run
$\boldsymbol{n}$	1	0
$\boldsymbol{n-1}$	0	1
$\boldsymbol{n-2}$	2^0	2^0
$\boldsymbol{n-3}$	2^1	2^1
…	…	…
2	2^{n-4}	2^{n-4}
1	2^{n-3}	2^{n-3}
Total	$\boldsymbol{2^{n-2}}$	$\boldsymbol{2^{n-2}}$
Grand total	$\boldsymbol{2^{n-1}}$	

The number of run sequences

Theorem 1.

The number l_n of cyclically distinct run sequences of length $2^n - 1$ is

$$l_n = \frac{1}{2^{n-2}} \binom{2^{n-2}}{2^{n-3}, 2^{n-4}, \ldots, 2^0, 1}^2 .$$

Corollary 1.

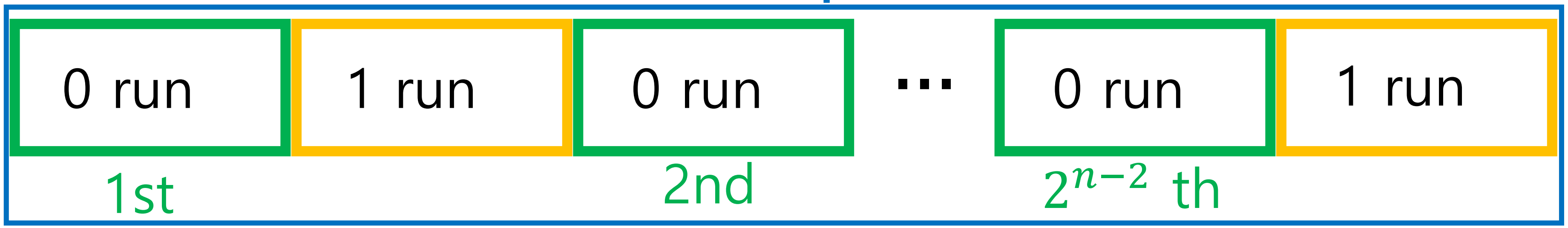
Let l_n be as defined in Theorem 1. Then

$$\frac{l_{n+1}}{l_n} = \frac{1}{2} \binom{2^{n-1}}{2^{n-2}}^2 \approx \frac{2}{\pi} 2^{2^n - n}.$$

Proof of Cor1. Use stirling's approximation [2].

Proof of Theorem 1

The run sequence

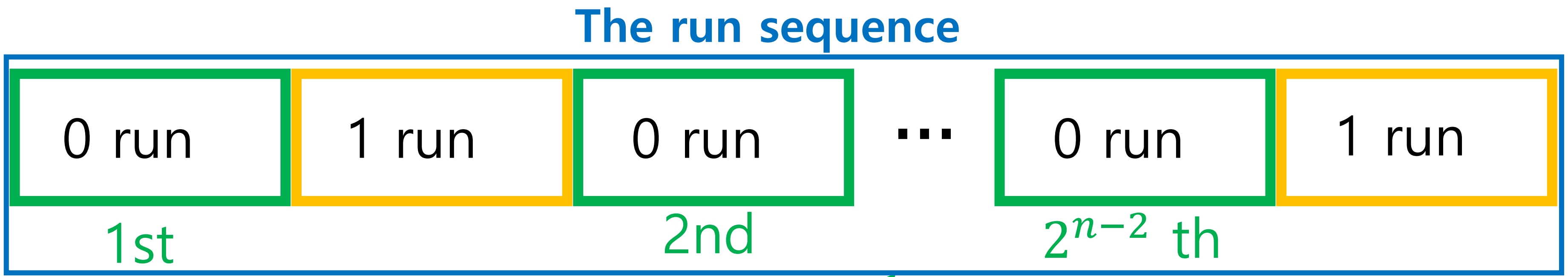


Length	# of 0's run
n	0
$n-1$	1
$n-2$	2^0
$n-3$	2^1
…	…
2	2^{n-4}
1	2^{n-3}
Total	$\mathbf{2^{n-2}}$

Fix the starting position as the unique run 0_{n-1}

※ 0_x: 0's run of length x

Proof of Theorem 1 (cont.)



Length	# of 0's run
n	0
$n-1$	1
$n-2$	2^0
$n-3$	2^1
…	…
2	2^{n-4}
1	2^{n-3}
Total	2^{n-2}

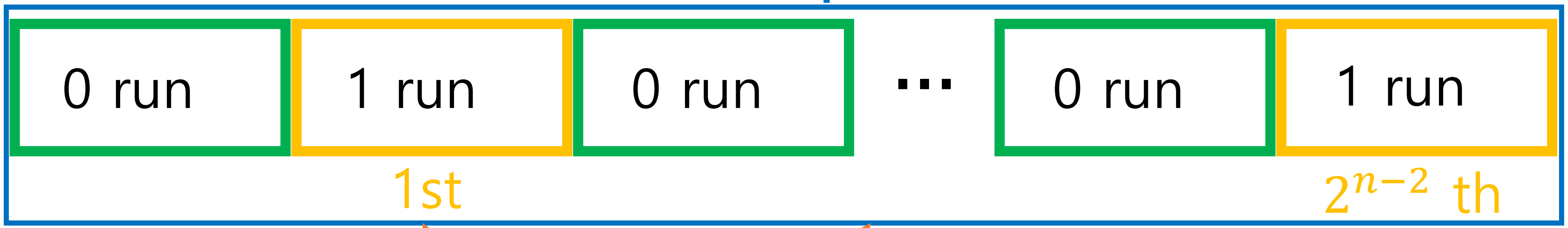
…

Calculate the number of permutations of the other **0's run**:

$$\binom{2^{n-2}-1}{2^{n-3}, 2^{n-4}, \ldots, 2^0}$$

Proof of Theorem 1 (cont.)

The run sequence



Length	# of 1's run
n	1
$n-1$	0
$n-2$	2^0
$n-3$	2^1
…	…
2	2^{n-4}
1	2^{n-3}
Total	2^{n-2}

…

Calculate the number of permutations of **1's run**:

$$\binom{2^{n-2}}{2^{n-3}, 2^{n-4}, \ldots, 2^0, 1}$$

Proof of Theorem 1 (cont.)

Product each number of permutations:

$$l_n = \binom{2^{n-2}-1}{2^{n-3}, 2^{n-4}, \ldots, 2^0} \binom{2^{n-2}}{2^{n-3}, 2^{n-4}, \ldots, 2^0, 1}$$

$$= \frac{1}{2^{n-2}} \binom{2^{n-2}}{2^{n-3}, 2^{n-4}, \ldots, 2^0, 1}^2$$

The number of run sequences

The number of **binary sequences of length 2^{n+1}-1** is about 2^{2^n} **times** of **the number of binary sequences of length 2^n-1 .**

Very similar for larger n

The number of **run sequences of length 2^{n+1}-1** is about $\frac{2}{\pi}2^{2^n-n}$ **times** of **the number of run sequences of length 2^n-1 .**

Ⅲ. The distribution of n-tuple vector in the run sequences of length $2^n - 1$

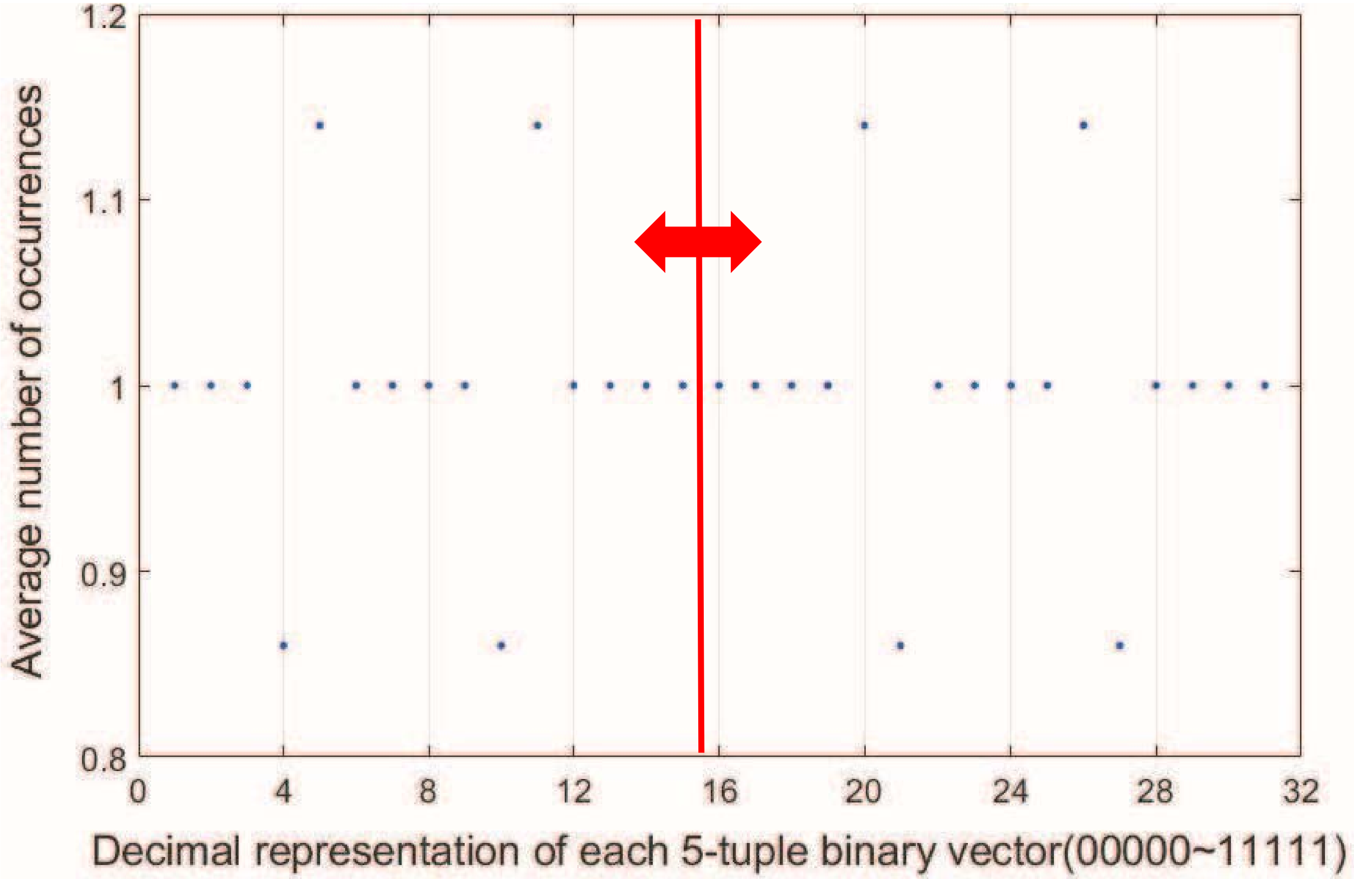
Span property

Definition 2. Span property [4], [5], [10]

A binary sequence of length $2^n - 1$ is said to have the **span property** if every n-tuple vector except for the all-zero vector occurs exactly once in one period. For simplicity, we call such a sequence **a span sequence.**

A span sequence is always **a run sequence**, but not conversely. In this talk, we investigate the $\boldsymbol{n}$**-tuple vector distribution** property of **run sequences** with or without **span property**

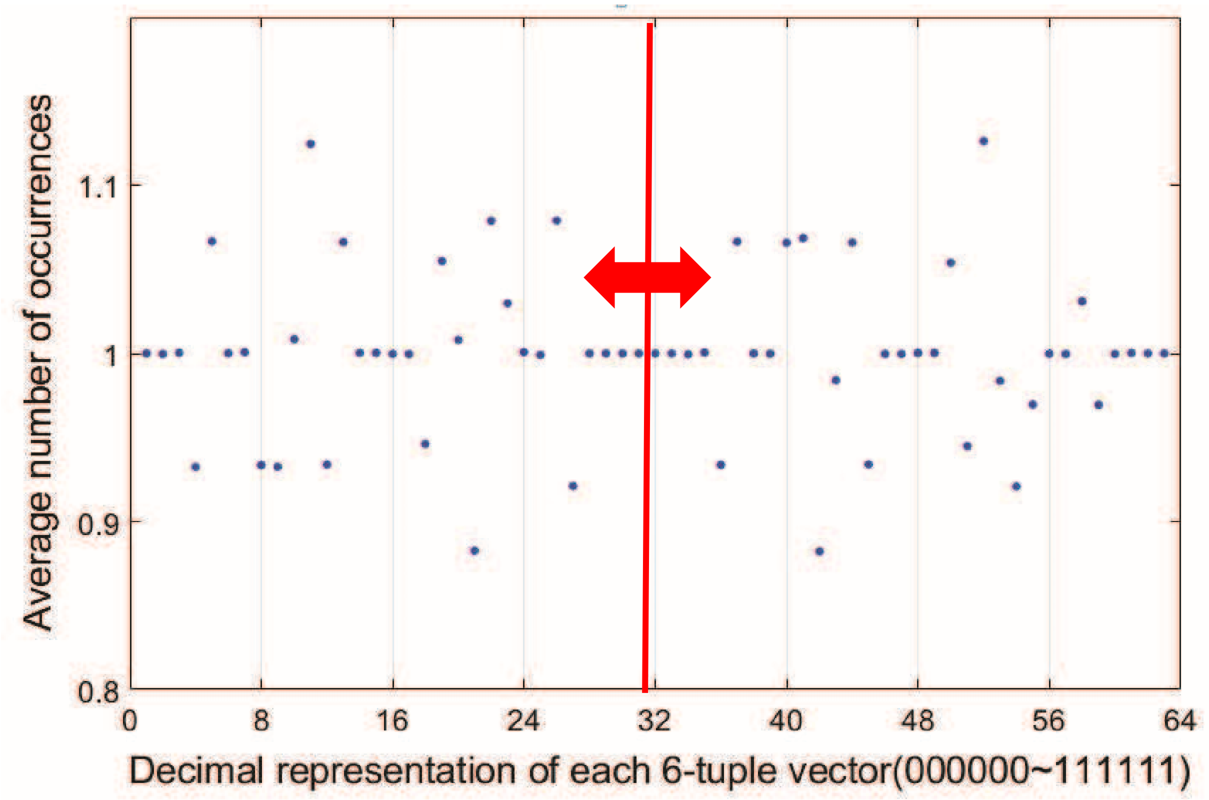
Average occurrence of n-tuple vector



Average occurrence of every **5** -tuple vector in all cyclically different run sequences of length **31**.

It looks symmetric. Is it exactly symmetric?

Average occurrence of n-tuple vector



Average occurrence of every $\mathbf{6}$-tuple vector in 1,000,000 cyclically different run sequences of length $\mathbf{63}$

It looks symmetric. Is it exactly symmetric?

Complement correspondence

Theorem 2.

For any run sequence $\boldsymbol{s_1}$ of length $2^n - 1$, there corresponds to another run sequence $\boldsymbol{s_2}$ such that

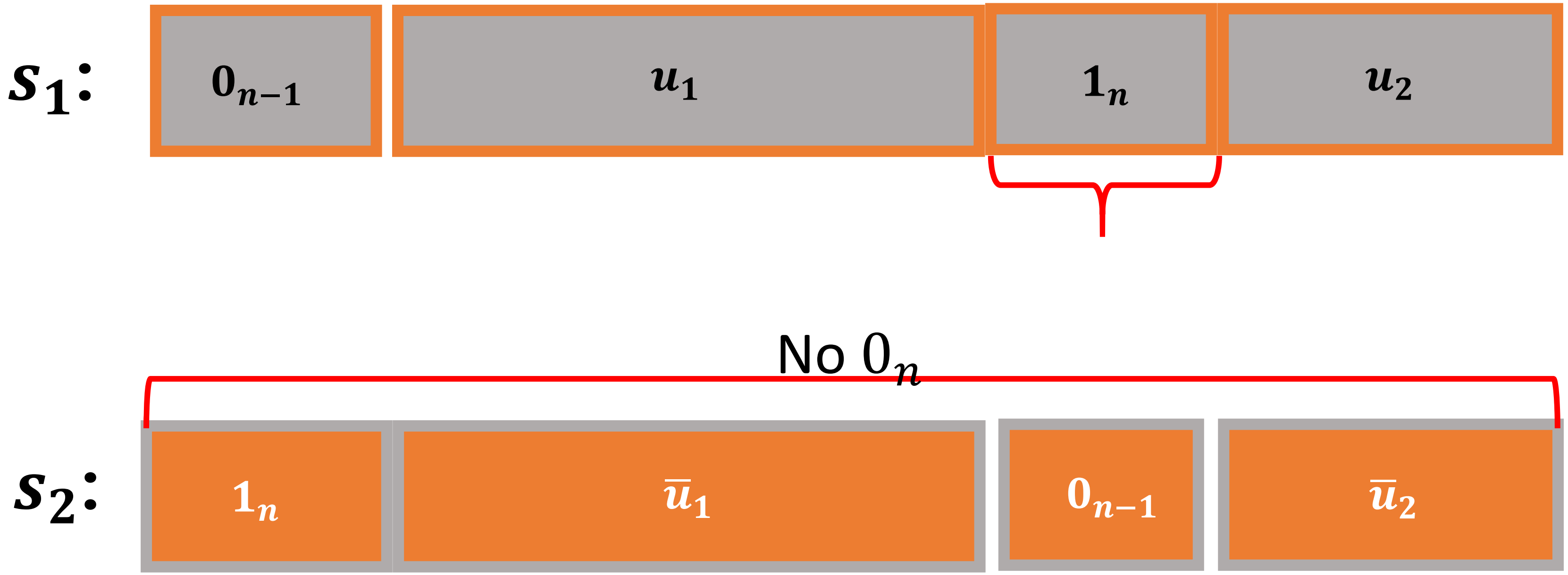
$$\begin{aligned}&\#\ of\ occurrences\ of\ v\ in\ \boldsymbol{s_1}\\ &= \#\ of\ occurrences\ of\ \bar{v}\ in\ \boldsymbol{s_2},\end{aligned}$$

for any n-tuple vector v except for the all-zero and the all-one vectors.

Theorem 2 implies that average occurrence of n-tuple vectors v and $\bar{v}$ are exactly same except for the all-zero and the all-one vectors.

Proof of Theorem 2

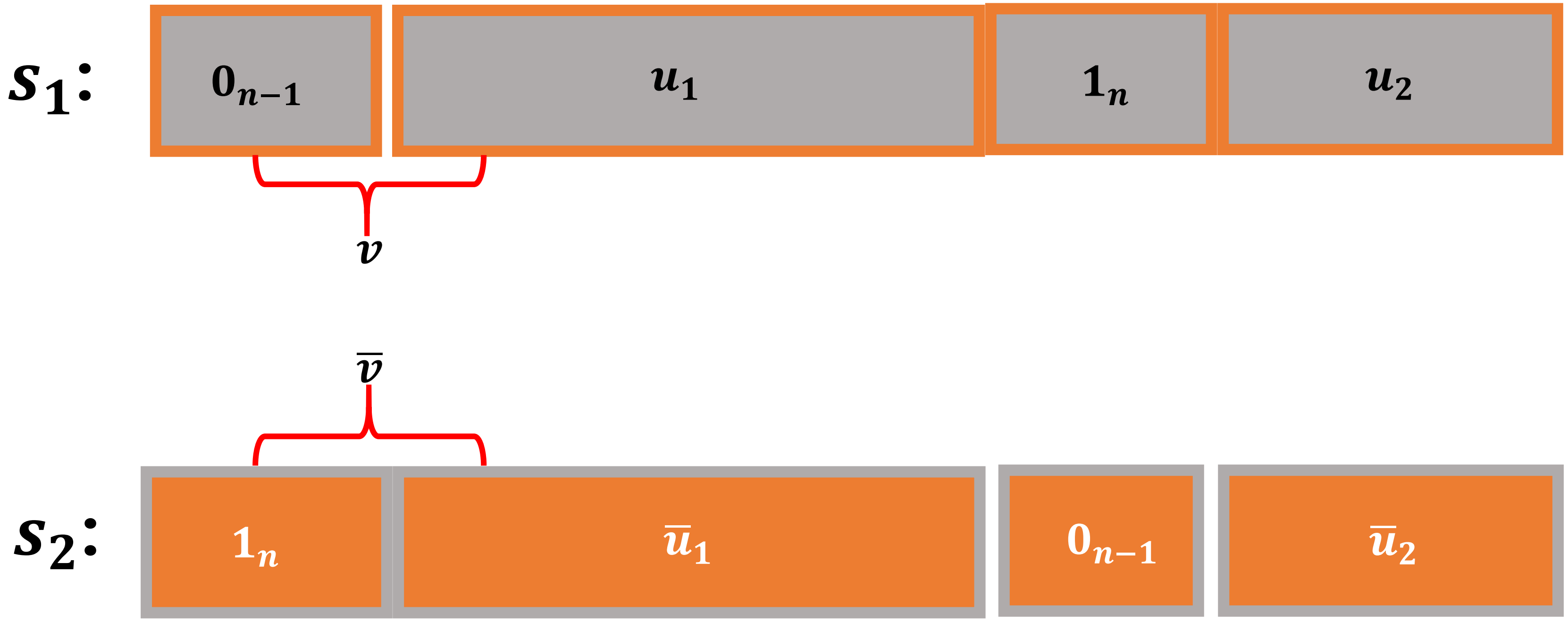
※ 0_x: 0's run of length x, 1_x: 1's run of length x



For the run sequence s_1, define s_2 as above. Obviously, s_2 is the run sequence and there is no 0_n(complement of 1_n) in s_2.

Proof of Theorem 2 (cont.)

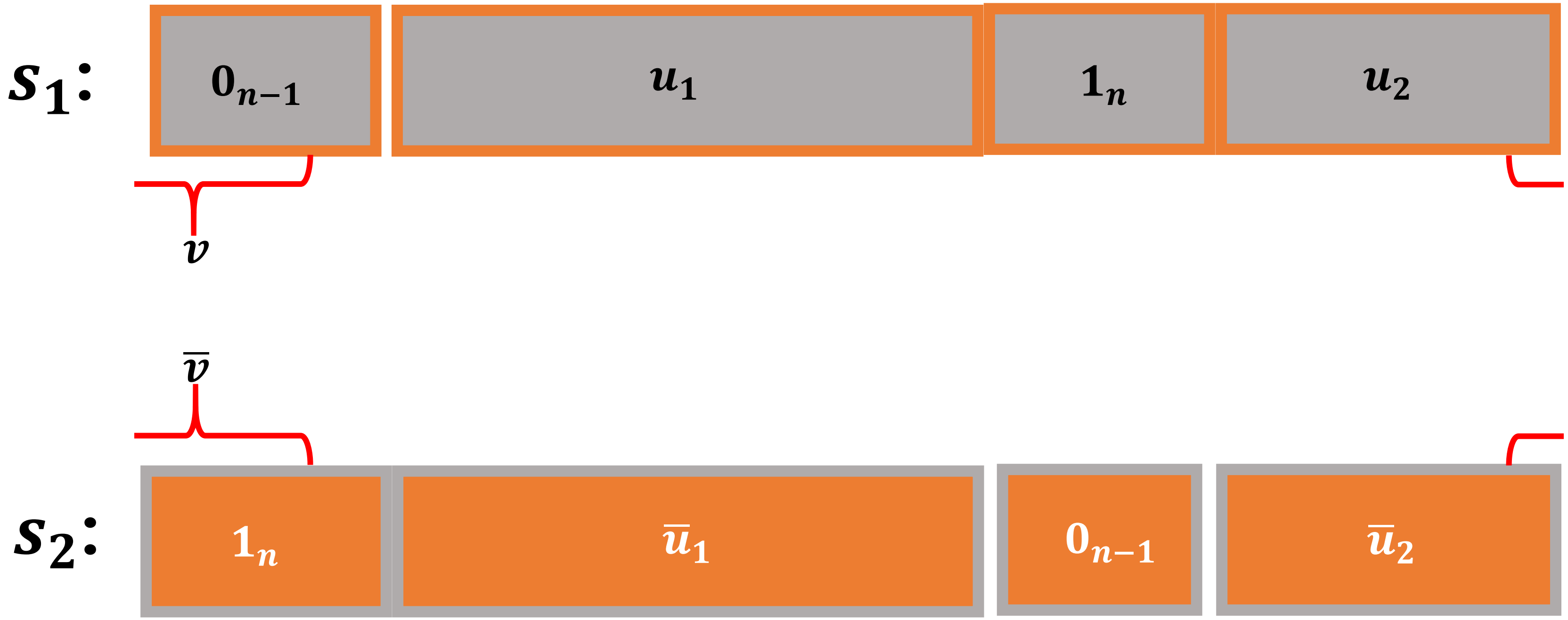
※ 0_x: 0's run of length x, 1_x: 1's run of length x



Except for the 1_n, any list v of adjacent n numbers in s_1 has corresponding list $\bar{v}$ of adjacent n numbers in s_2

Proof of Theorem 2 (cont.)

※ 0_x: 0's run of length x, 1_x: 1's run of length x



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Proof of Theorem 2 (cont.)

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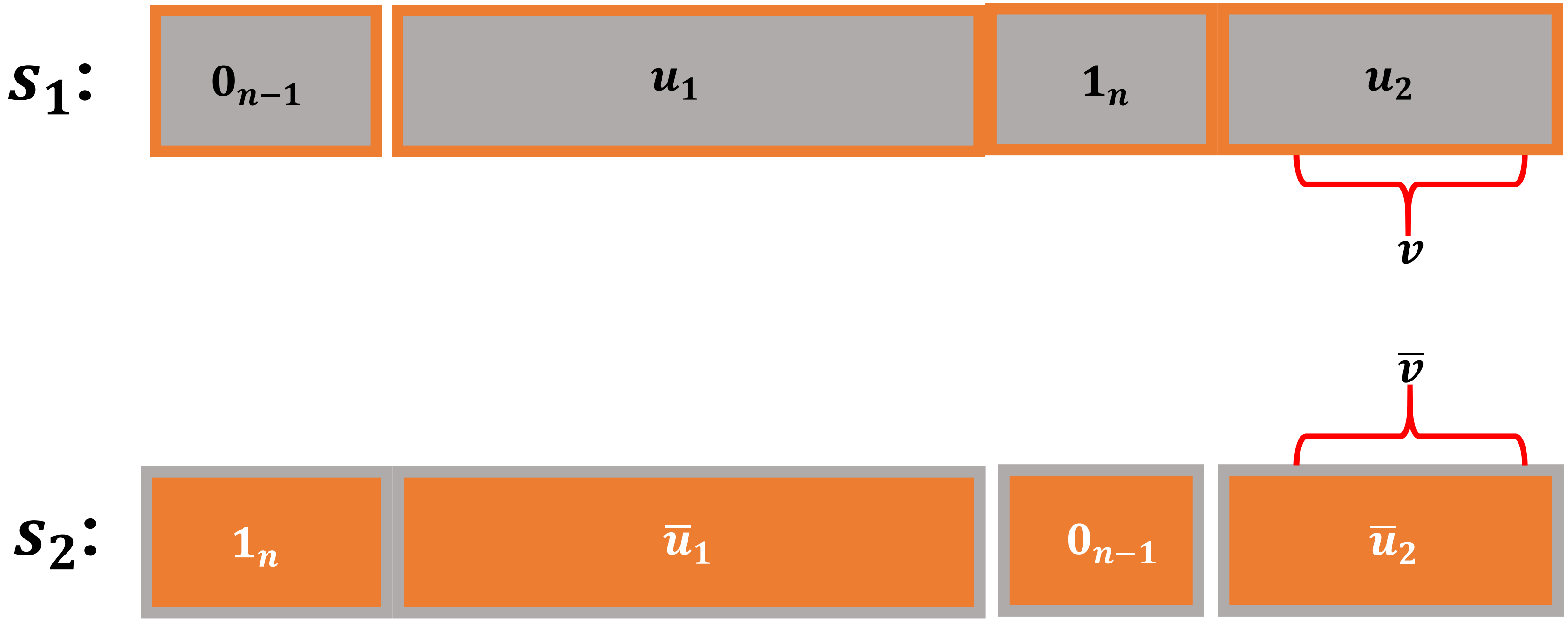

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Proof of Theorem 2 (cont.)

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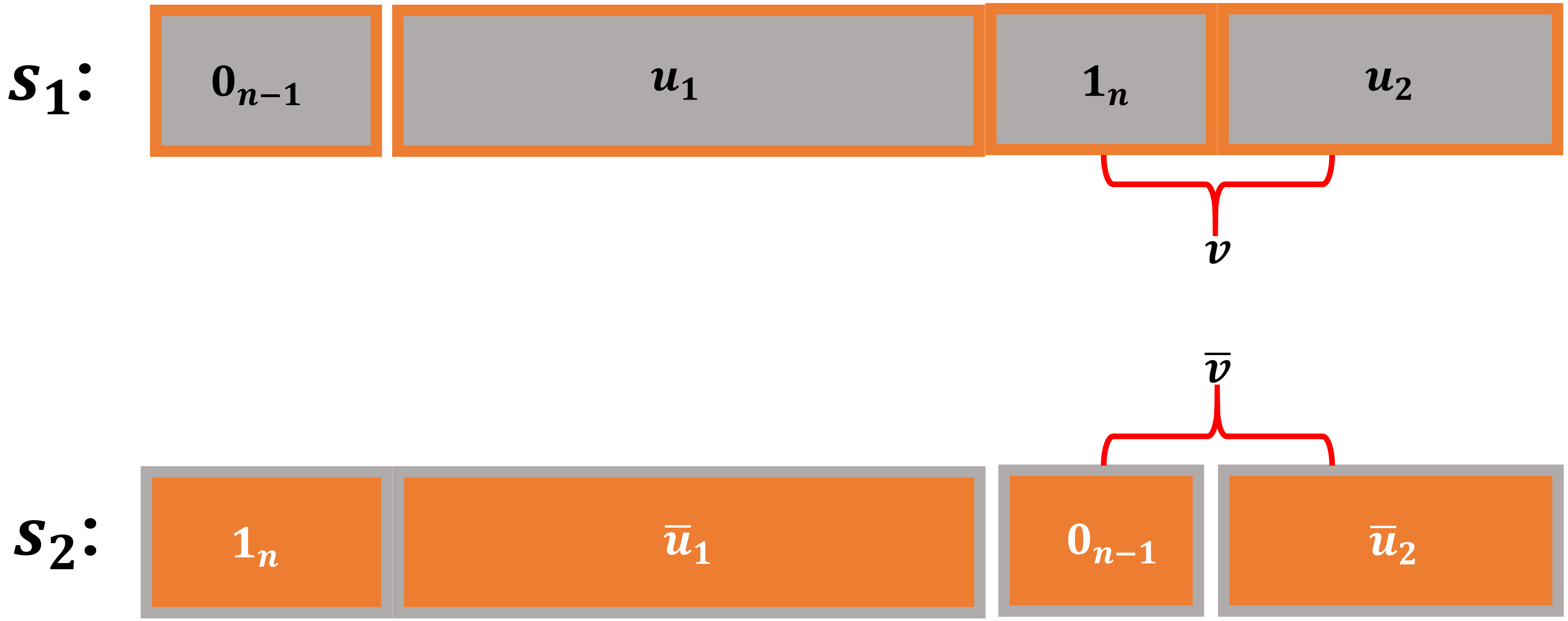

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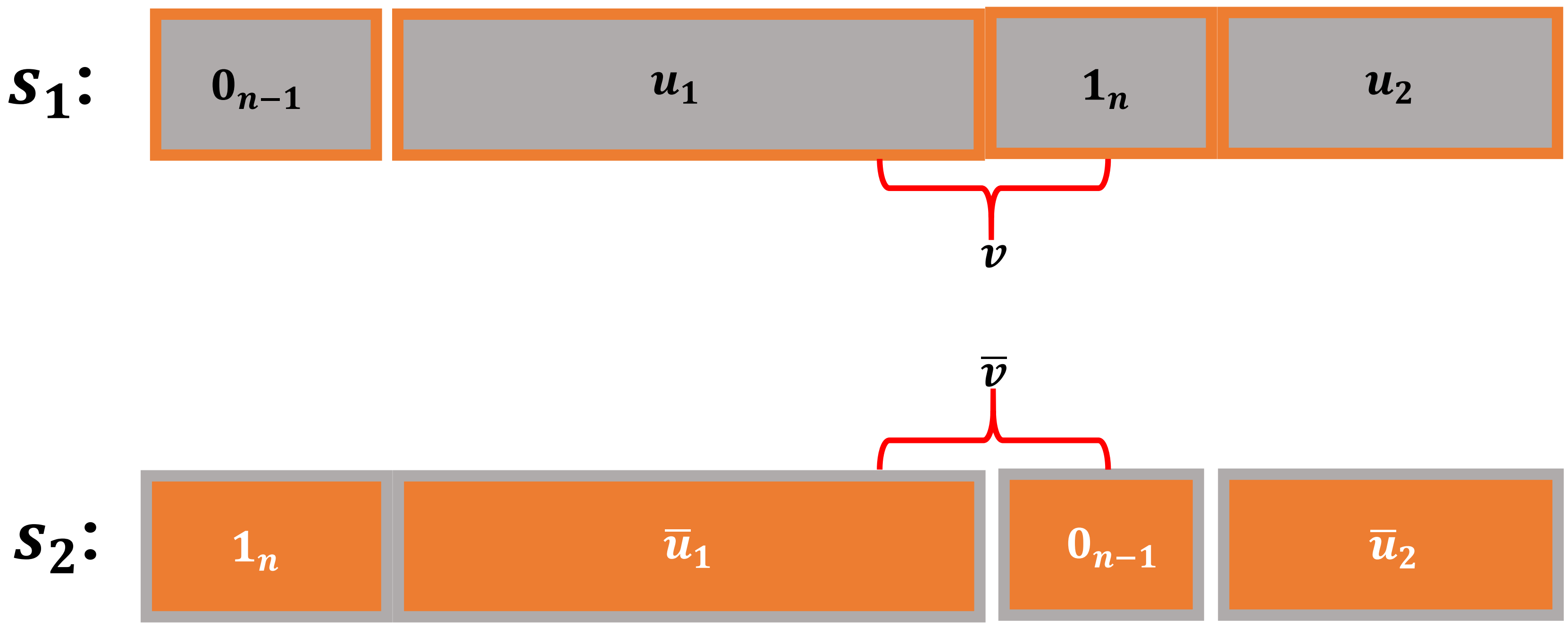
Proof of Theorem 2 (cont.)

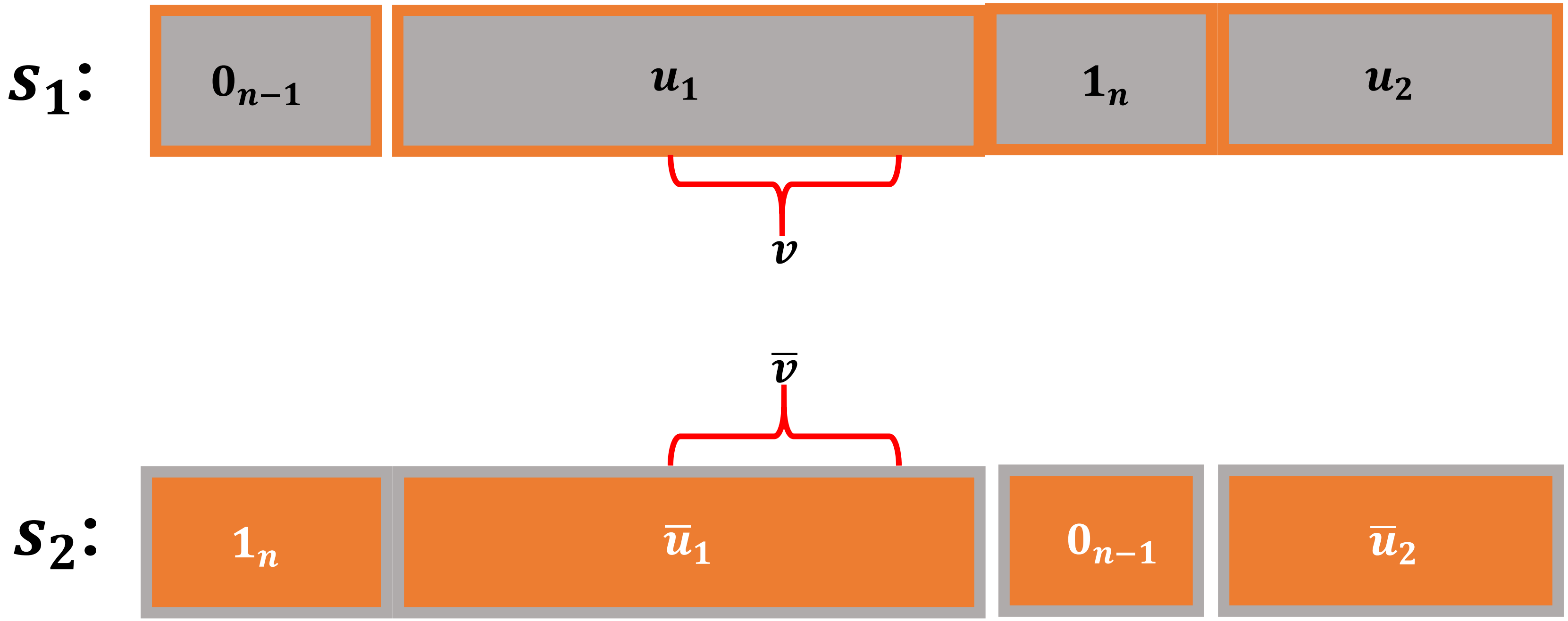
※ 0_x: 0's run of length x, 1_x: 1's run of length x



Except for the 1_n, any list v of adjacent n numbers in s_1 has corresponding list $\overline{v}$ of adjacent n numbers in s_2

Proof of Theorem 2 (cont.)

※ 0_x: 0's run of length x, 1_x: 1's run of length x



Except for the 1_n, any list v of adjacent n numbers in s_1 has corresponding list $\overline{v}$ of adjacent n numbers in s_2

Example of Theorem 2

The following two sequenes of length 31 are examples of Theorem 2:

s_1: <u>0 0 0 0</u> 1 1 0 0 1 0 <u>1 1 1 1 1</u> 0 1 1 0 1 1 1 0 1 0 0 0 1 0 0 1

01000 01000

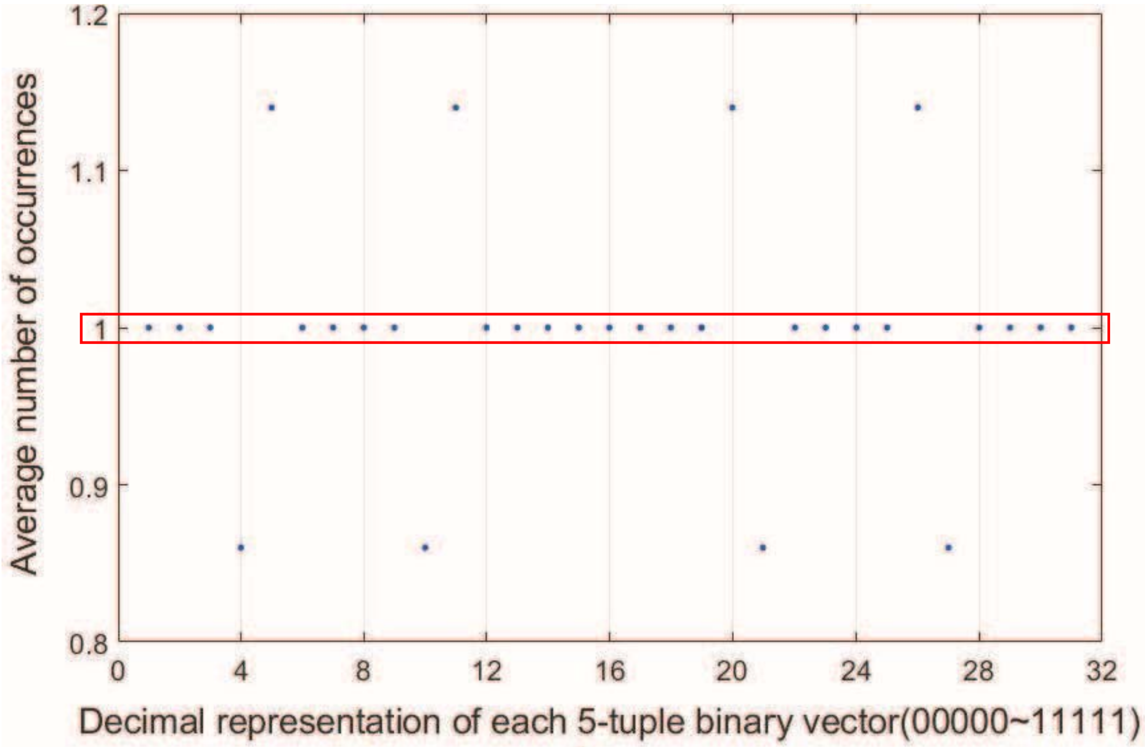
10111 10111

s_2: <u>1 1 1 1 1</u> 0 0 1 1 0 1 <u>0 0 0 0</u> 1 0 0 1 0 0 0 1 0 1 1 1 0 1 1 0

Example of Theorem 2 (cont.)

v: 5-tuple vector	# of v in s_1	# of v in s_2	$\overline{v}$: complement of v	# of $\overline{v}$ in s_1	# of $\overline{v}$ in s_2
00000	0	0	11111	1	1
00001	1	1	11110	1	1
00010	1	2	11101	2	1
00011	1	0	11100	0	1
00100	2	2	11011	2	2
00101	1	1	11010	1	1
00110	1	1	11001	1	1
00111	0	0	11000	0	0
01000	2	2	10111	2	2
01001	1	1	10110	1	1
01010	0	0	10101	0	0
01011	1	1	10100	1	1
01100	1	0	10011	0	1
01101	1	2	10010	2	1
01110	1	1	10001	1	1
01111	1	1	10000	1	1

Average occurrence of n-tuple vector

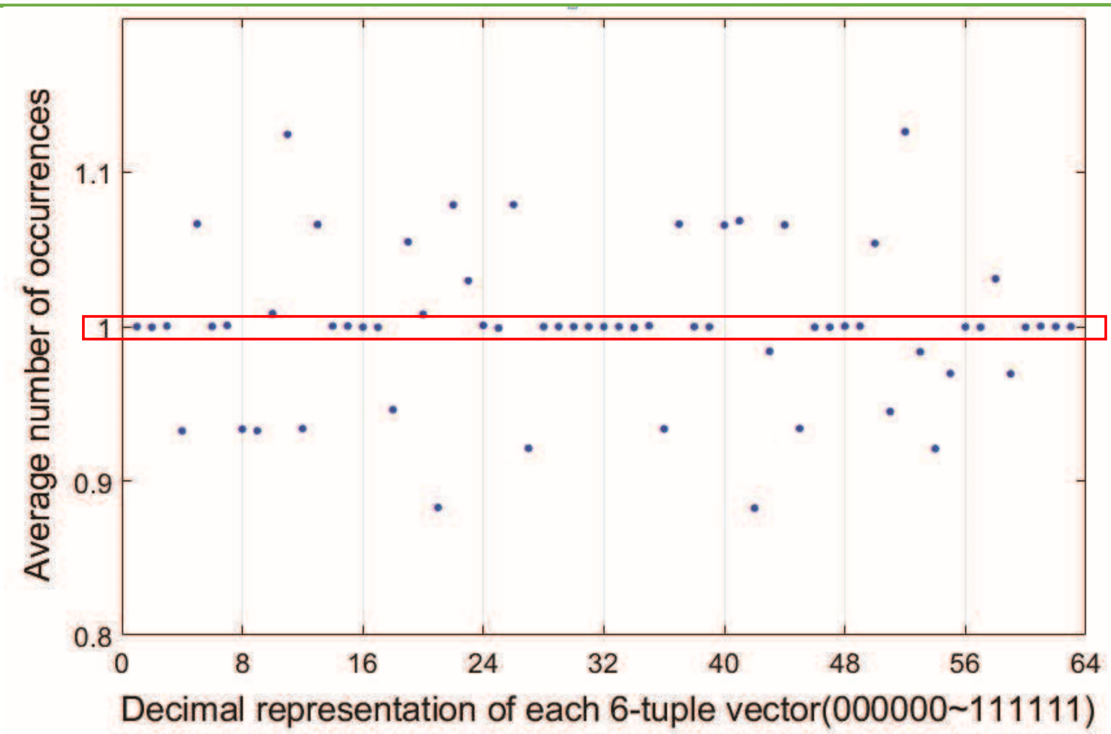


Average occurrence of every **5** -tuple vector in all cyclically different run sequences of length **31**.

It is surprising that there exist some n-tuple vectors such that the average number is equal to 1.

Which n-tuple vectors have an average of 1?

Average occurrence of n-tuple vector



Average occurrence of every **6** -tuple vector in 1,000,000 cyclically different run sequences of length **63**.

It is surprising that there exist some n-tuple vectors such that the average number is equal to 1.

Which n-tuple vectors have an average of 1?

The vectors which occur exactly once

Theorem 3.

The following 7 n-tuple vectors **occur exactly once** in any run sequence of length $2^n - 1$:

$$a0_{n-2}b \quad \text{and} \quad \mathrm{a}1_{n-2}b,$$

where $a, b \in \{0,1\}$, except for the all-zero vector.

Theorem 3 is the subcase of average of 1.

The vectors whose average is one

Theorem 4.

The average number of occurrences of the following n-tuple vectors in all the run sequence of length $2^n - 1$ **is equal to 1** :

$$a0_k 1_{n-2-k} b \quad \text{and} \quad \mathrm{a}1_k 0_{n-2-k} b,$$

where $a, b \in \{0,1\}$ and $k = 0, 1, \dots, n-2,$ except for the all-zero vector.

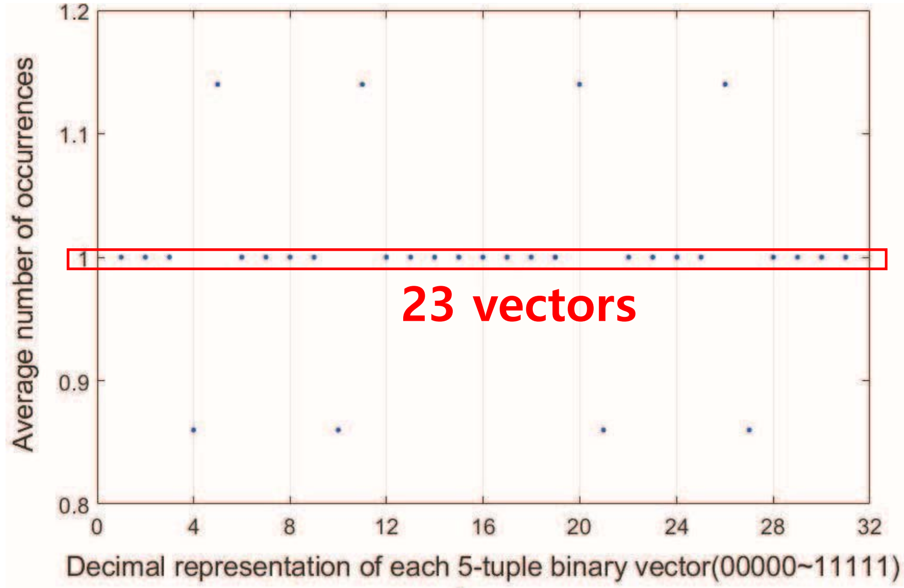
Example of Theorem 3, 4

23 vectors

$n = 5$

00011, 00111, 11100, 11000,
00010, 00110, 11101, 11001,
10011, 10111, 01100, 01000,
10010, 10110, 01101, 01001
in Theorem 4.

00001, 10000, 10001, 01110,
01111, 11110, 11111
in Theorem 3, 4.



The 23 vectors described in Theorem 4 cover all the 5-tuple vectors whose average number of 1.

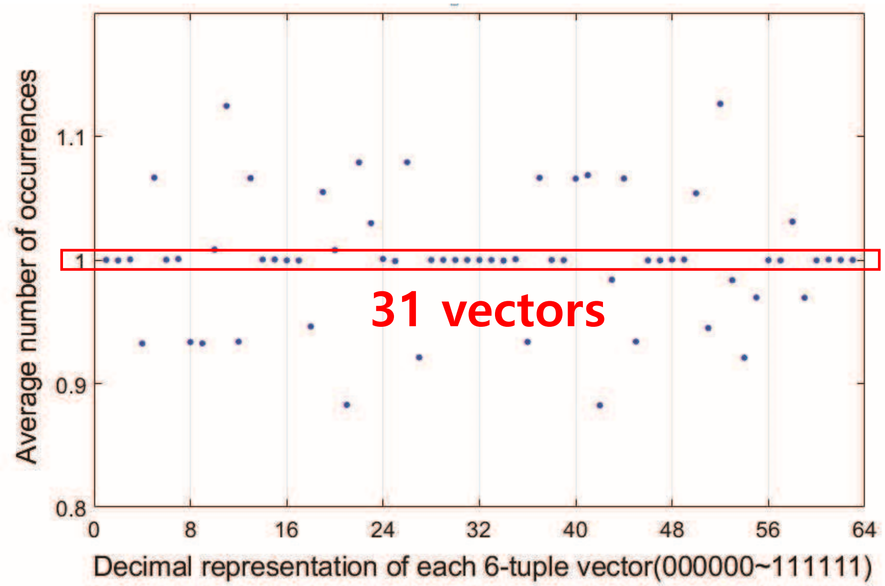
Example of Theorem 3, 4 (cont.)

$n = 6$

31 vectors

000011, 000111, 001111, 111100, 111000, 110000, 000010, 000110, 001110, 111101, 111001, 110001, 100011, 100111, 101111, 011100, 011000, 010000, 100010, 100110, 10110, 011101, 011001, 010001 in Theorem 4.

000001, 100000, 100001, 011110, 011111, 111110, 111111 in Theorem 3, 4.



The 31 vectors described in Theorem 4 cover all the 6-tuple vectors whose average number of 1.

IV. Conclusion

Conclusion

- Caluculate **the number of run sequences** of length $2^n - 1$ and approximate **increase rate of that number** by an exponential form of 2
- Present some interesting properties about **n-tuple vector distribution of run sequences** of $2^n - 1$ **.**

Future Works

- Study **n-tuple vector distribution of run sequences** of length 2^n **for deBruijn sequences.**
- Develop **generating method for run, span and deBruijn sequences**

Thank you

References

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Appendix-Theorem3

Theorem 3.

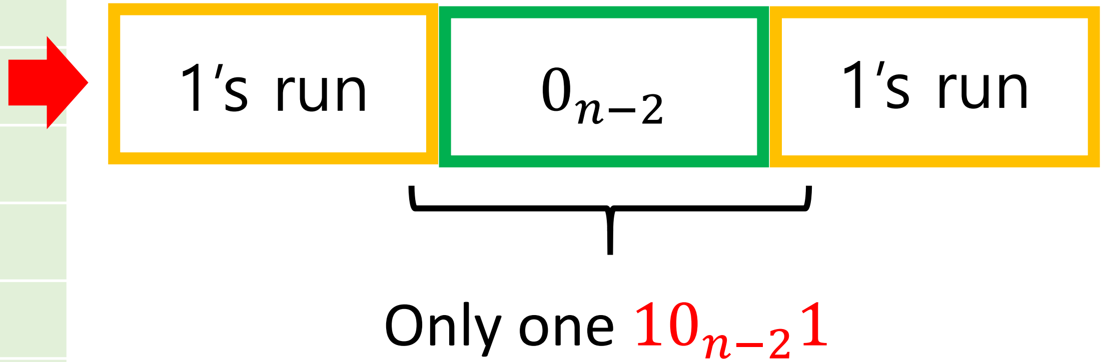
The following 7 n-tuple vectors **occur exactly once** in any run sequence of length $2^n - 1$:

$$a0_{n-2}b \quad \text{and} \quad \text{a}1_{n-2}b,$$

where $a, b \in \{0,1\}$, except for the all-zero vector.

Appendix: Proof of Theorem 3

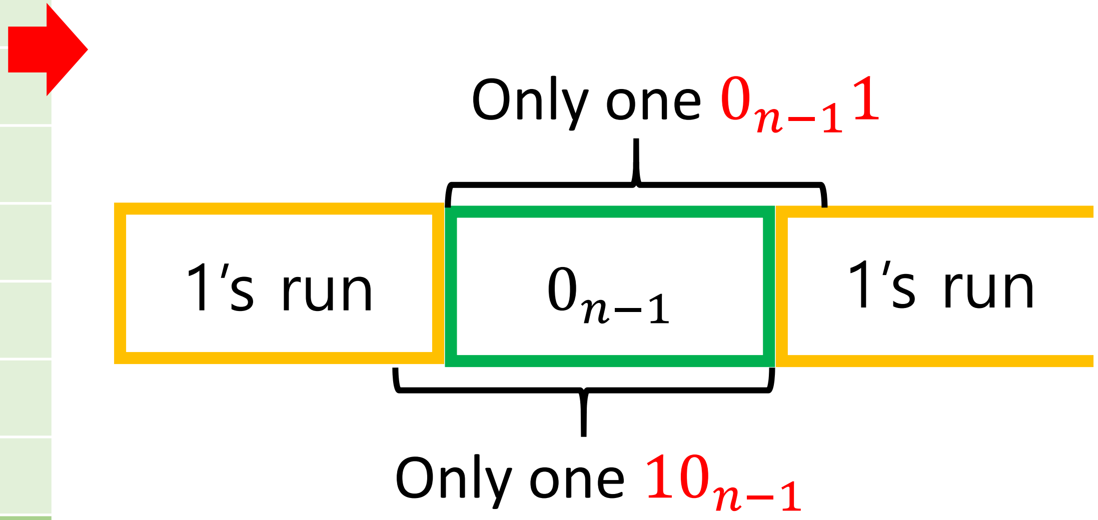
Length	0's run
$\boldsymbol{n}$	0
$\boldsymbol{n-1}$	1
$\boldsymbol{n-2}$	2^0
$\boldsymbol{n-3}$	2^1
…	…
$\mathbf{2}$	2^{n-4}
$\mathbf{1}$	2^{n-3}
Total	$\boldsymbol{2^{n-2}}$



An n-tuple vector $\mathbf{10_{n-2}1}$ occurs exactly once.

Appendix: Proof of Theorem 3 (cont.)

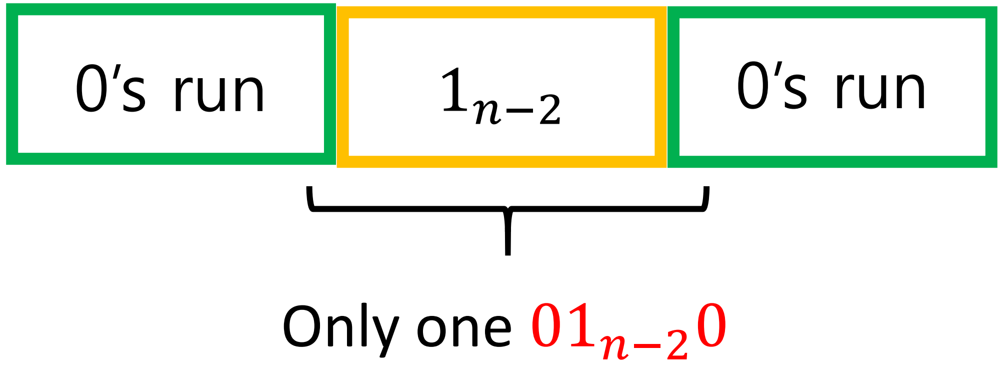
Length	0's run
n	0
$n-1$	1
$n-2$	2^0
$n-3$	2^1
…	…
2	2^{n-4}
1	2^{n-3}
Total	2^{n-2}



Two n-tuple vector $\mathbf{10_{n-1}}$ and $\mathbf{0_{n-1}1}$ occur exactly once.

Appendix: Proof of Theorem 3 (cont.)

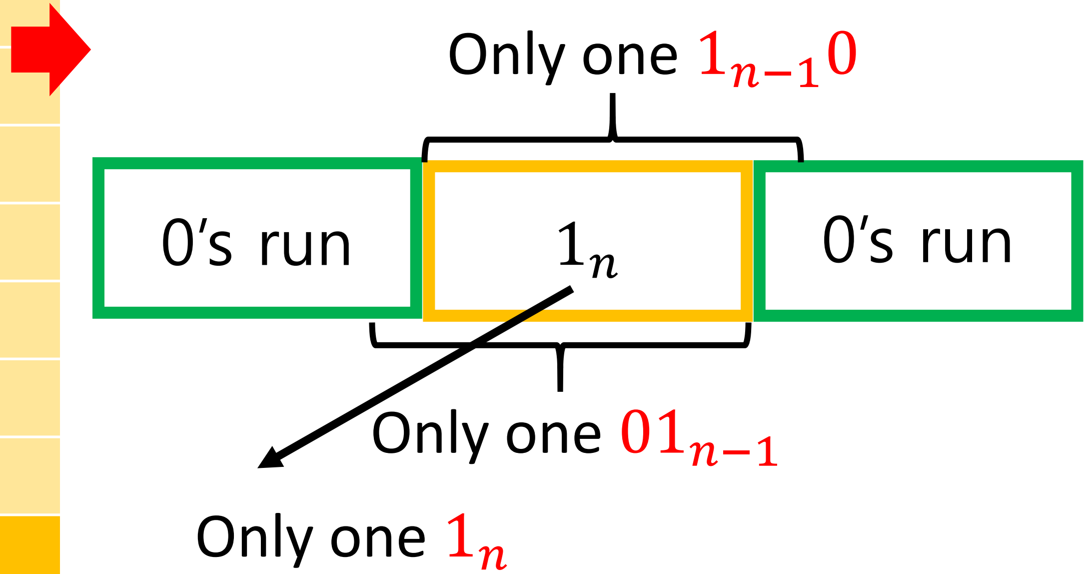
Length	1's run
n	1
$n-1$	0
$n-2$	2^0
$n-3$	2^1
…	…
2	2^{n-4}
1	2^{n-3}
Total	$\mathbf{2^{n-2}}$



An n-tuple vector $\mathbf{01_{n-2}0}$ occurs exactly once.

Appendix: Proof of Theorem 3 (cont.)

Length	1's run
n	1
$n-1$	0
$n-2$	2^0
$n-3$	2^1
…	…
2	2^{n-4}
1	2^{n-3}
Total	2^{n-2}



Three n-tuple vector $\mathbf{1_{n-1}0}$, $\mathbf{01_{n-1}}$, and $\mathbf{1_n}$ occur exactly once.

Appendix: Theorem 4

Theorem 4.

The average number of occurrences of the following n-tuple vectors in all the run sequence of length $2^n - 1$ **is equal to 1** :

$$a0_k 1_{n-2-k} b \quad \text{and} \quad \mathrm{a}1_k 0_{n-2-k} b,$$

where $a, b \in \{0,1\}$ and $k = 0, 1, \dots, n-2,$ except for the all-zero vector.

Appendix: Proof of Theorem 4

The run sequence

0_{n-1}	1's run	⋯	0_x	1_y	⋯	0's run	1's run
1st	1st		k-th	k-th		2^{n-2} th	2^{n-2} th

Length	1's run	0's run
n	1	0
$n-1$	0	1
$n-2$	2^0	2^0
…	…	…
x	No matter	2^{n-2-x}
y	2^{n-2-y}	No matter
…	…	…
2	2^{n-4}	2^{n-4}
1	2^{n-3}	2^{n-3}
Total	2^{n-2}	2^{n-2}

For $1 \le x, y \le n-2$,

# of run sequences such that
k-th 0's run is 0_x and
k-th 1's run is 1_y

$$= \binom{2^{n-2}-2}{2^{n-3}, \ldots, 2^{n-2-x}-1, \ldots, 2^0} \times \binom{2^{n-2}-1}{2^{n-3}, \ldots, 2^{n-2-y}-1, \ldots, 2^0, 1}$$

Appendix: Proof of Theorem 4(cont.)

The run sequence

0_{n-1}	1's run	⋯	0_x	1_y	⋯	0's run	1's run
1st	1st		k-th	k-th		2^{n-2} th	2^{n-2} th

Since k is possible for $k = 2, 3, \dots, 2^{n-2}$, # of occurrences of 10_x1_y0 in all the run sesquences is $(2^{n-2} - 1)$ times ☐ **:**

$$= \binom{2^{n-2}-1}{2^{n-3}, \dots, 2^{n-2-x}-1, \dots, 2^0, 1} \times \binom{2^{n-2}-1}{2^{n-3}, \dots, 2^{n-2-y}-1, \dots, 2^0, 1}$$

For $1 \le x, y \le n-2$,

\# of run sequences such that
k-th 0's run is 0_x and
k-th 1's run is 1_y

$$= \binom{2^{n-2}-2}{2^{n-3}, \dots, 2^{n-2-x}-1, \dots, 2^0} \times \binom{2^{n-2}-1}{2^{n-3}, \dots, 2^{n-2-y}-1, \dots, 2^0, 1}$$

Appendix: Proof of Theorem 4(cont.)

For $1 \le x, y \le n-2$,

$$T(x,y) = \binom{2^{n-2}-1}{2^{n-3}, \dots, 2^{n-2-x}-1, \dots, 2^0, 1} \times \binom{2^{n-2}-1}{2^{n-3}, \dots, 2^{n-2-y}-1, \dots, 2^0, 1}$$

$$T(n-1,y) = \binom{2^{n-2}-1}{2^{n-3}, \dots, 2^0} \times \binom{2^{n-2}-1}{2^{n-3}, \dots, 2^{n-2-y}-1, \dots, 2^0, 1}$$

$$T(x,n) = \binom{2^{n-2}-1}{2^{n-3}, \dots, 2^{n-2-x}-1, \dots, 2^0, 1} \times \binom{2^{n-2}-1}{2^{n-3}, \dots, 2^0}$$

$$T(n-1,n) = \binom{2^{n-2}-1}{2^{n-3}, \dots, 2^0}^2$$

※ $T(a,b)$: The number of occurrences of 10_a1_b0

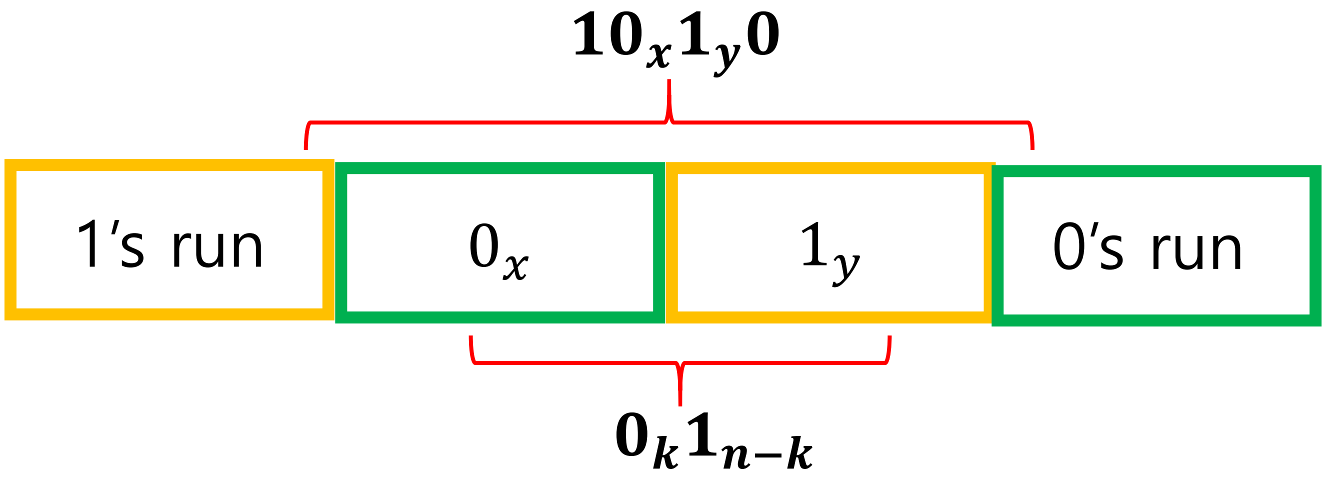
Appendix: Proof of Theorem 4(cont.)

We prove only two cases for $2 \leq k \leq n-2$:
Case 1: the n-tuple vector $0_k 1_{n-k}$
Case 2: the n-tuple vector $1 0_{k-1} 1_{n-k-1} 0$

Appendix: Proof of Theorem 4(cont.)

Case 1: the n-tuple vector $0_k 1_{n-k}$

The n-tuple vector $\mathbf{0_k 1_{n-k}}$ **can be a part of** $\mathbf{10_x 1_y 0}$ for $k \le x \le n-1$ and $n-k \le y \le n-2$or $y = n$.



Appendix: Proof of Theorem 4(cont.)

Case 1: the n-tuple vector $0_k 1_{n-k}$

The n-tuple vector $\mathbf{0_k 1_{n-k}}$ **can be a part of** $\mathbf{10_x 1_y 0}$ for $k \leq x \leq n-1$ and $\mathrm{n}-k \leq y \leq n-2$or $y = n$.

Therefore **the total number of occurrences of** $\mathbf{0_k 1_{n-k}}$ in all cyclically distinct run sequences of length $2^n - 1$ **is equal to**

$$\sum_{x=k}^{n-1} \sum_{y=n-k}^{n} T(x, y).$$

※ $T(a, b)$: The number of occurrences of $10_a 1_b 0$

Appendix: Proof of Theorem 4(cont.)

Case 1: the n-tuple vector $0_k 1_{n-k}$

$$\sum_{x=k}^{n-1}\sum_{y=n-k}^{n} T(x,y) = \left\{\binom{2^{n-2}-1}{2^{n-3},\dots,2^0} + \sum_{x=k}^{n-2}\binom{2^{n-2}-1}{2^{n-3},\dots,2^{n-2-x}-1,\dots,2^0,1}\right\}$$

$$\times\left\{\binom{2^{n-2}-1}{2^{n-3},\dots,2^0} + \sum_{y=n-k}^{n-2}\binom{2^{n-2}-1}{2^{n-3},\dots,2^{n-2-y}-1,\dots,2^0,1}\right\}$$

$$= \left(\frac{1}{2^{n-2}} + \sum_{x=k}^{n-2}\frac{2^{n-x-2}}{2^{n-2}}\right)\left(\frac{1}{2^{n-2}} + \sum_{y=n-k}^{n-2}\frac{2^{n-y-2}}{2^{n-2}}\right)\binom{2^{n-2}}{2^{n-3},\dots,2^0,1}^2$$

$$= \frac{1}{2^{n-2}}\binom{2^{n-2}}{2^{n-3},\dots,2^0,1}^2 = \boldsymbol{l_n}$$

※ $T(a,b)$: The number of occurrences of 10_a1_b0

Appendix: Proof of Theorem 4(cont.)

Case 2: the n-tuple vector $10_{k-1}1_{n-k-1}0$

The total number of occurrences of $\mathbf{10_{k-1}1_{n-k-1}0}$ **is equal to**

$$T(k-1, n-k-1) = \binom{2^{n-2}-1}{2^{n-3}, \dots, 2^{n-1-k}-1, \dots, 2^0, 1} \times \binom{2^{n-2}-1}{2^{n-3}, \dots, 2^{k-1}-1, \dots, 2^0, 1}$$
$$= \frac{2^{n-1-k}}{2^{n-2}} \cdot \frac{2^{k-1}}{2^{n-2}} \binom{2^{n-2}}{2^{n-3}, \dots, 2^0, 1}^2 = \frac{1}{2^{n-2}} \binom{2^{n-2}}{2^{n-3}, \dots, 2^0, 1}^2 = \boldsymbol{l_n}.$$

※ $T(a, b)$: The number of occurrences of 10_a1_b0