Zero-Correlation-Zone Sonar Sequences

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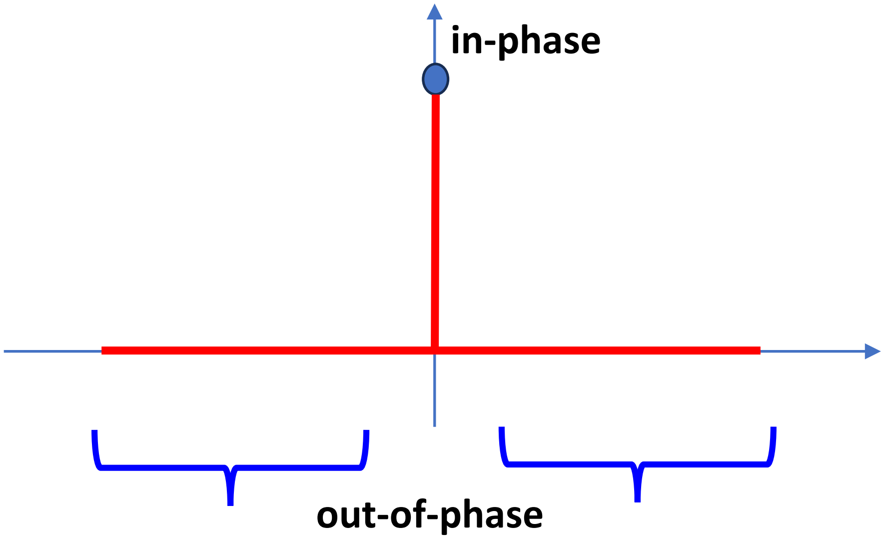
Binary sequences with PERFECT autocorrelation

- PERFECT autocorrelation:

$$C(\tau) = n \quad \text{when} \quad \tau = 0 \text{ (in-phase)}$$

$$C(\tau) = 0 \quad \text{when} \quad \tau \neq 0 \text{ (out-of-phase)}$$



- We know that

  it is impossible for the length > 4

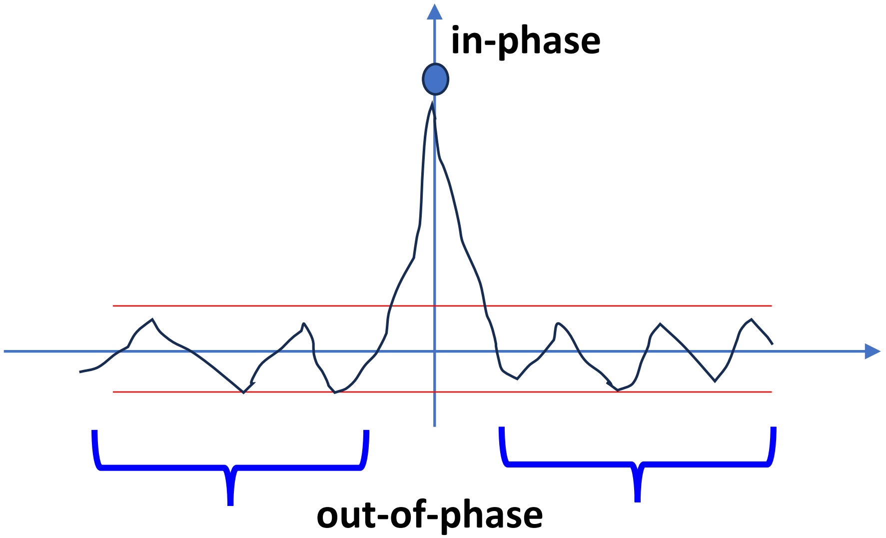
Binary sequences with ideal autocorrelation

- GOOD (IDEAL) autocorrelation:

$$C(\tau) = n \quad \text{when} \quad \tau = 0$$

$$|\boldsymbol{C}(\boldsymbol{\tau})| \leq \mathbf{1} \quad \textbf{when} \quad \boldsymbol{\tau} \neq \mathbf{0}$$

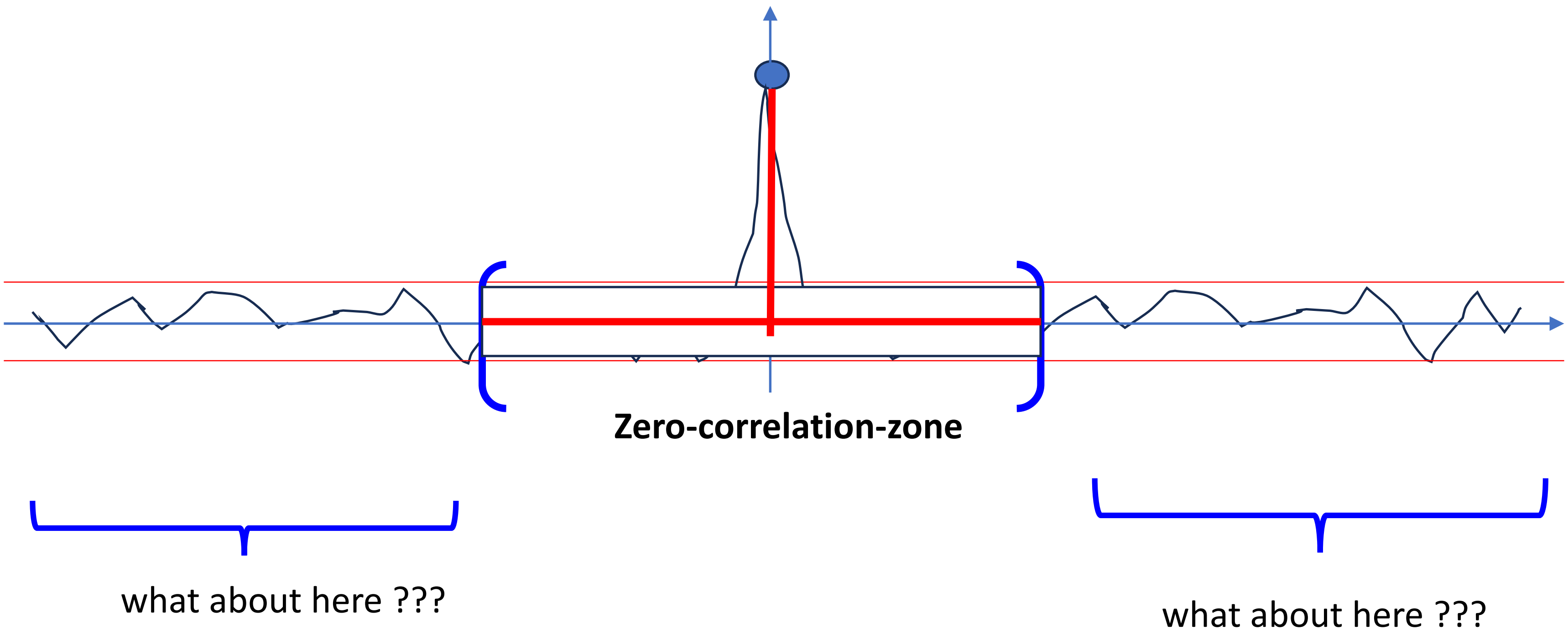
- Examples
  - ✓ m-sequences
  - ✓ quadratic residue sequences
  - ✓ and many others



sidelobe is not PERFECT in general

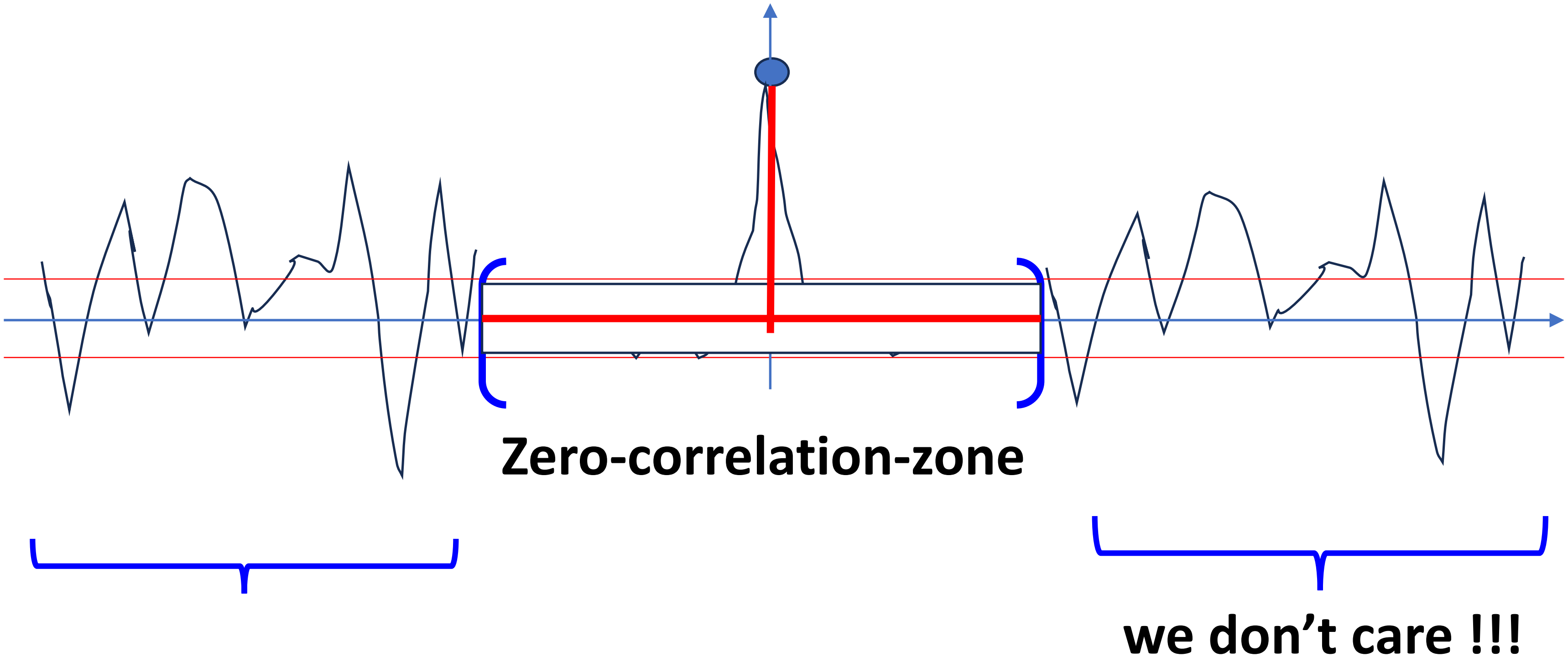
Binary sequences with **Zero-correlation-zone** (ZCZ)

P. Z. Fan, et al, "Class of binary sequences with zero correlation zone," Electronics Letters, vol. 35, no. 10, pp. 777-779, May. 1999.



Typical binary sequences with ZCZ

P. Z. Fan, et al, "Class of binary sequences with zero correlation zone," **Electronics Letters**, vol. 35, no. 10, pp. 777-779, May. 1999.



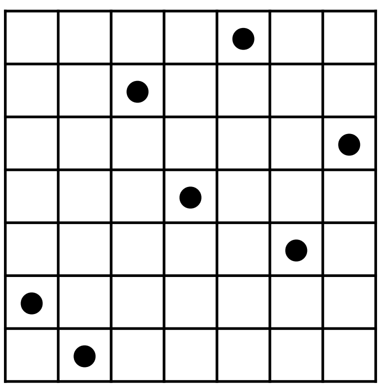
Two-dimensional versions

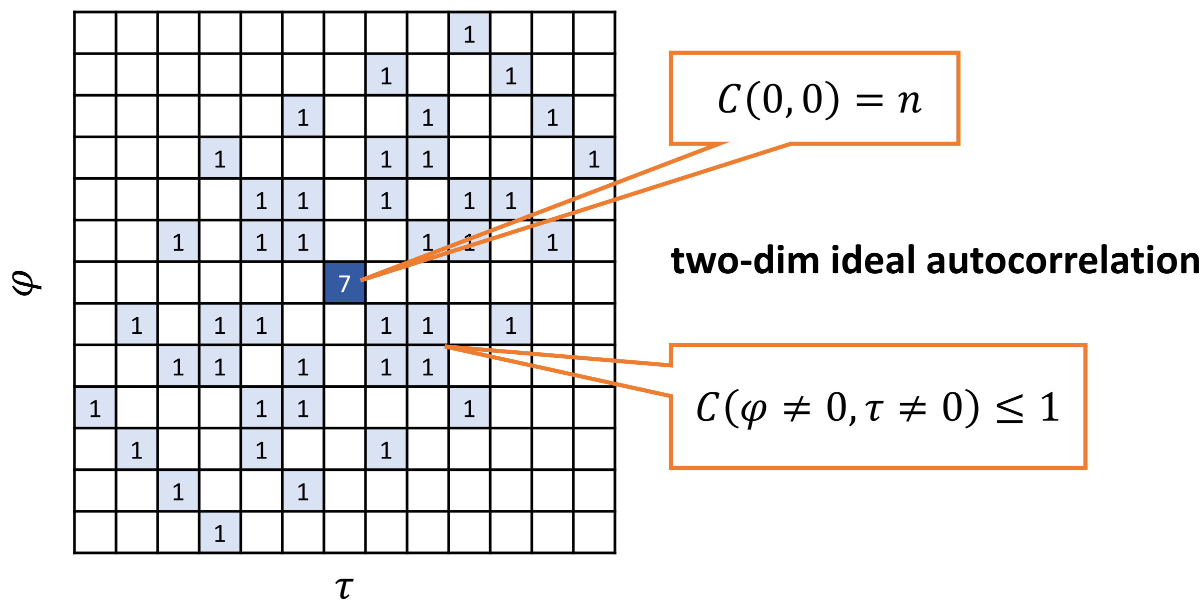
SONAR sequences

- Two-dimensional synchronizing patterns of dots and blanks with minimal ambiguity.
- Active sonar systems (and also for pulse compression radar)

  -- improve target detection performance.

S. W. Golomb and **H. Taylor**, "Two-Dimensional Synchronization Patterns for Minimum Ambiguity," **IEEE Transactions on Information Theory**, vol. 28, no. 4, pp. 600–604, Jul. 1982.



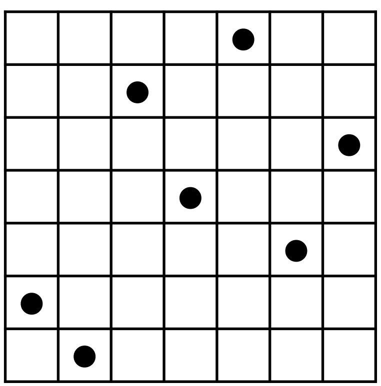


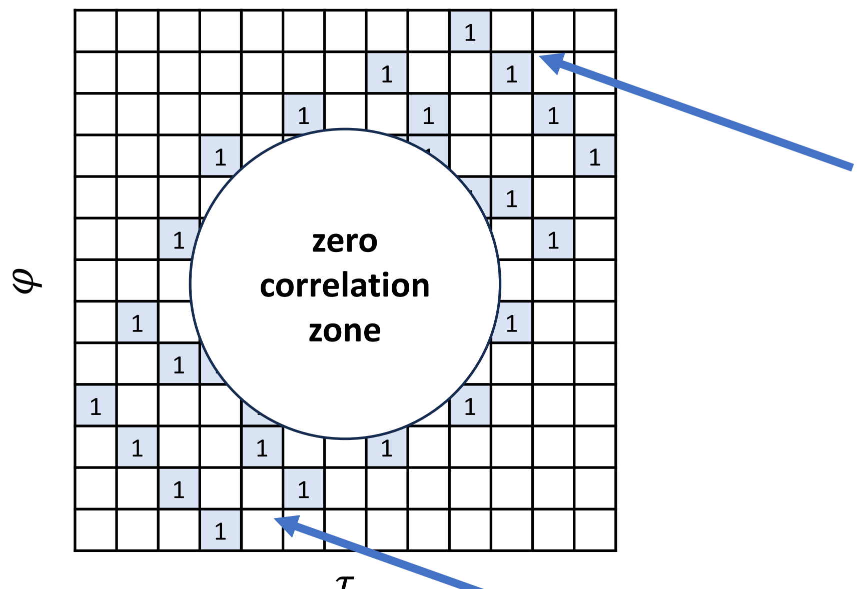
Two-dimensional versions

main contribution of this talk

SONAR sequences with ZCZ

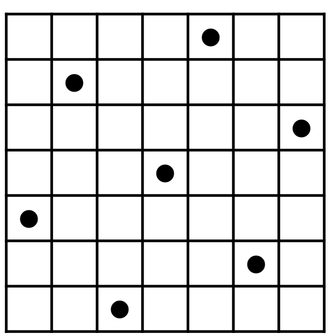
what about the other area outside ZCZ ???

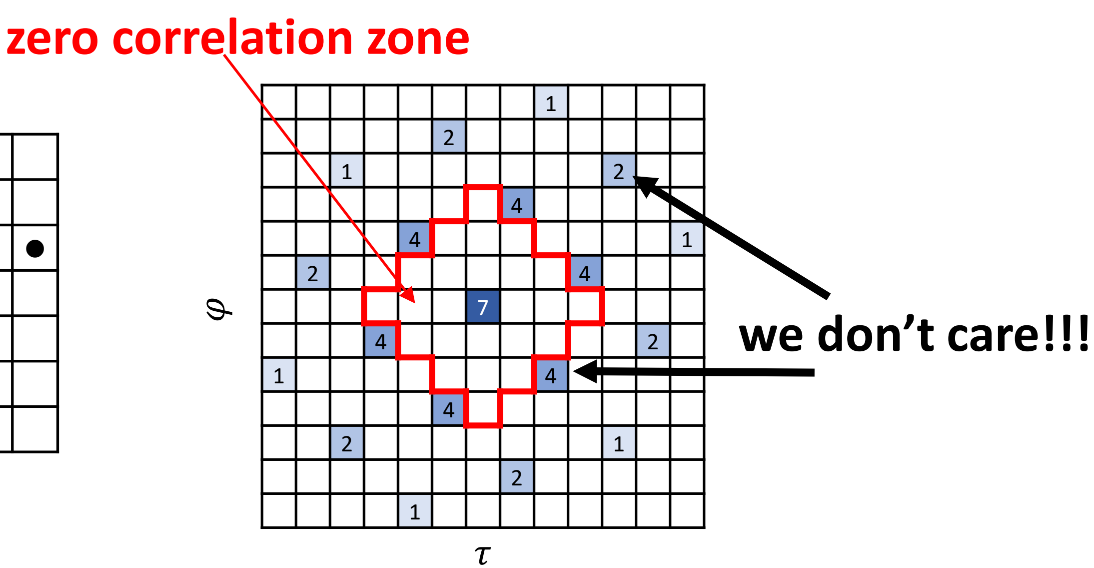




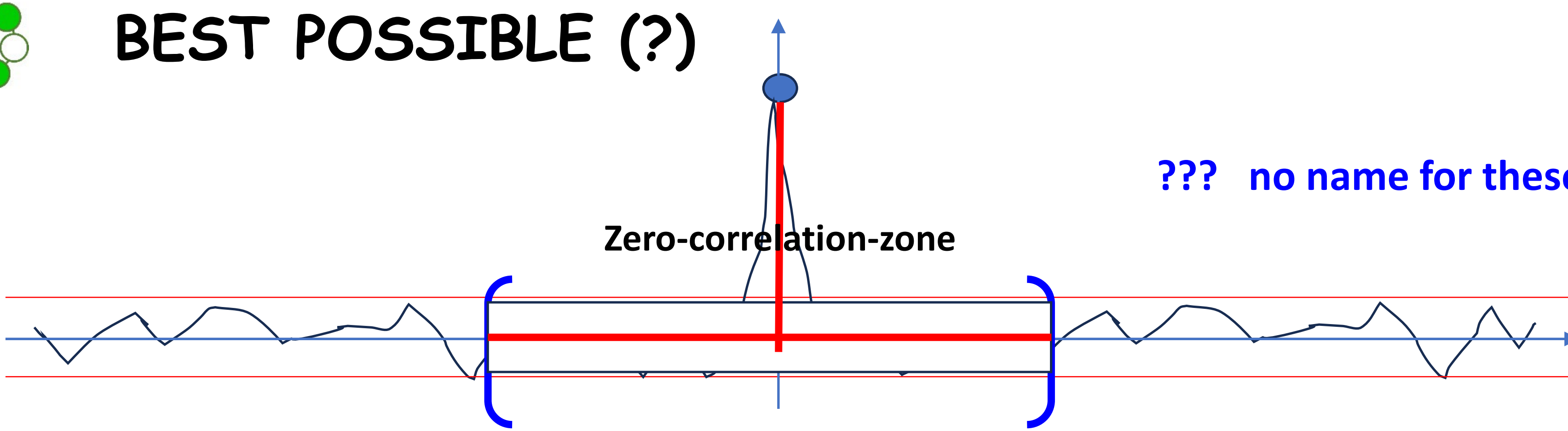
Two-dimensional versions

Typical SONAR sequences with ZCZ





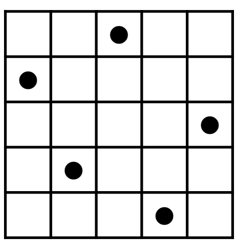
BEST POSSIBLE (?)

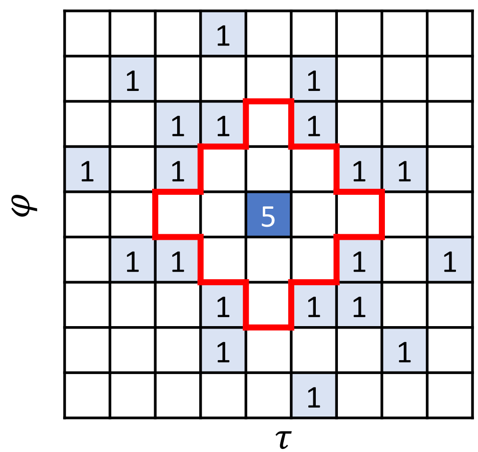


??? no name for these

Binary sequences with a large ZCZ

still satisfying the ideal autocorrelation property outside ZCZ





ZCZ-DD sonar sequences

ZCZ sonar sequences

still satisfying the sonar sequence property outside ZCZ

Sonar sequences

- A function $f:\{1,2,\ldots,n\} \to \{1,2,\ldots,m\}$ has the **distinct difference (DD) property** if

$$f(u+h) - f(u), \quad \text{for all possible } h \text{ and } u, \text{ are all distinct.}$$

- An (m,n) **sonar sequence** is a function $f:\{1,2,\ldots,n\} \to \{1,2,\ldots,m\}$ with the DD property.

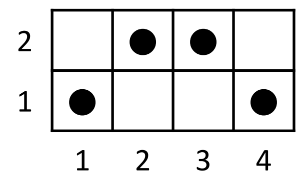
- An (m,n) sonar sequence is **optimal** if n is the largest possible for given m.

Example

An optimal $(2,4)$ sonar sequence

$[1,2,2,1]$

2×4 sonar array

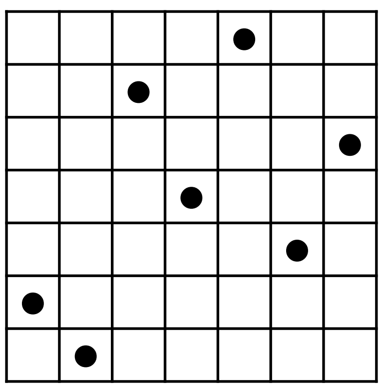


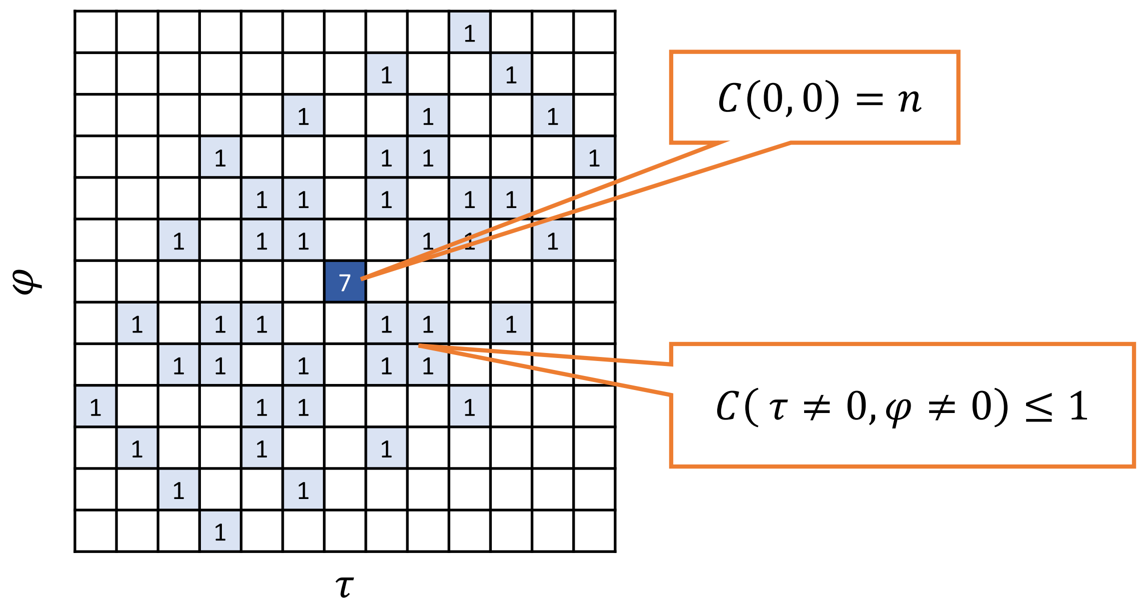
[11] S. W. Golomb and H. Taylor, "Two-Dimensional Synchronization Patterns for Minimum Ambiguity," IEEE Transactions on Information Theory, vol. 28, no. 4, pp. 600–604, Jul. 1982.

Correlation of sonar sequence [12]

- The discrete non-periodic autocorrelation $C(\tau, \varphi)$ is the number of coincidences between dots in a sonar array $A(i, j)$ and its shift $A(i+\tau, j+\varphi)$.

Example



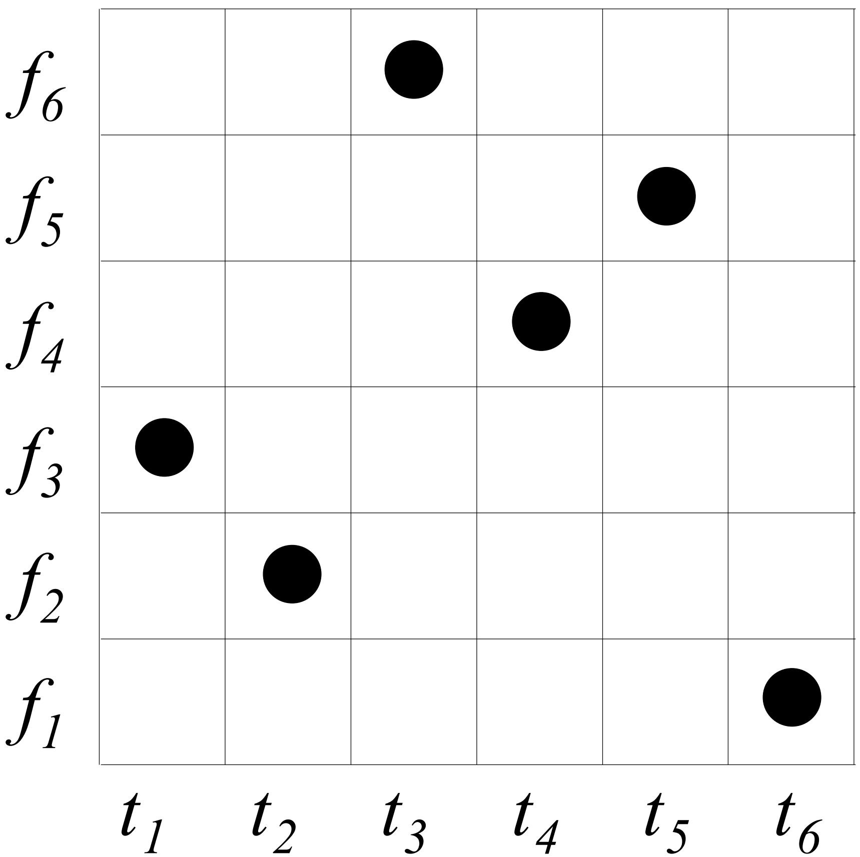


[12] S. W. Golomb and H. Taylor, “Constructions and properties of Costas arrays,” Proceedings of the IEEE, vol. 72, no. 9, pp. 1143–1163, Sep. 1984.

Auticorrelation $C(\tau,\varphi)$

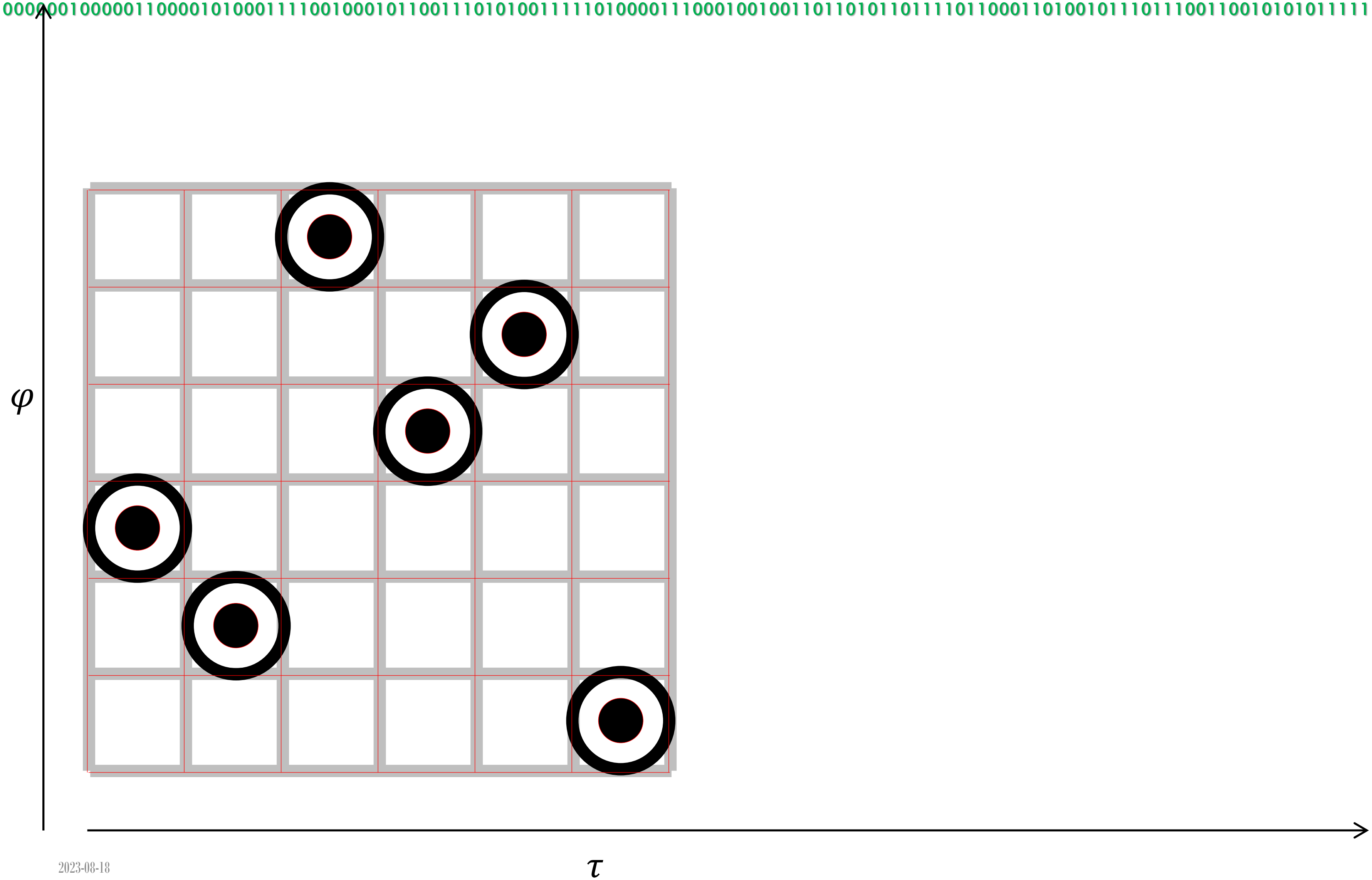
1000000100000110000101000111100100010110011101010011111010000111000100100110110101101111011000110100101110111001100101010111111

a (6,6) sonar array



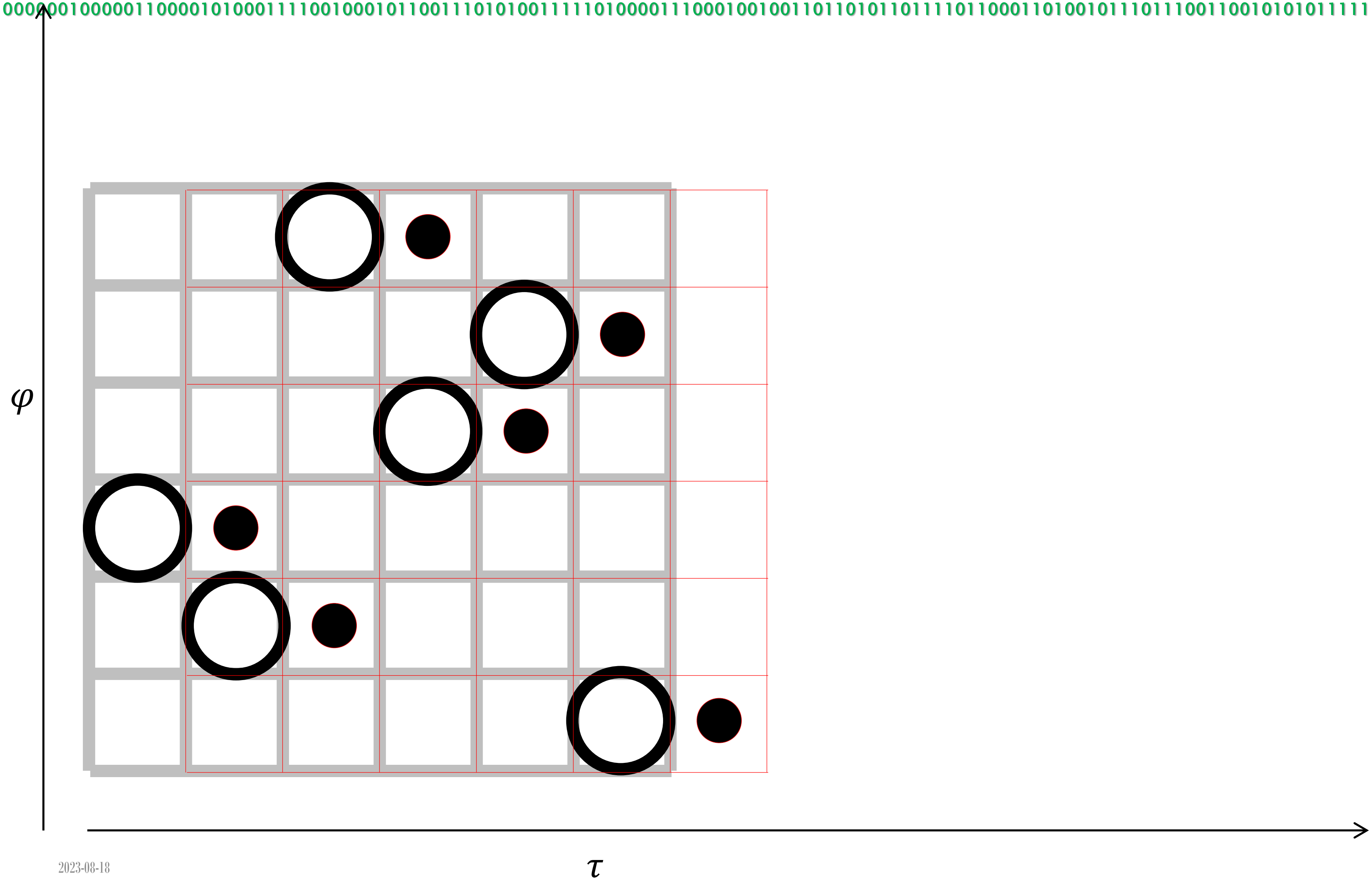
Autocorrelation at (τ=0, φ=0)

10000001000000110000101000111100100010110011101010011111010000111000100100110110101101111011000110100101110111001100101010111111



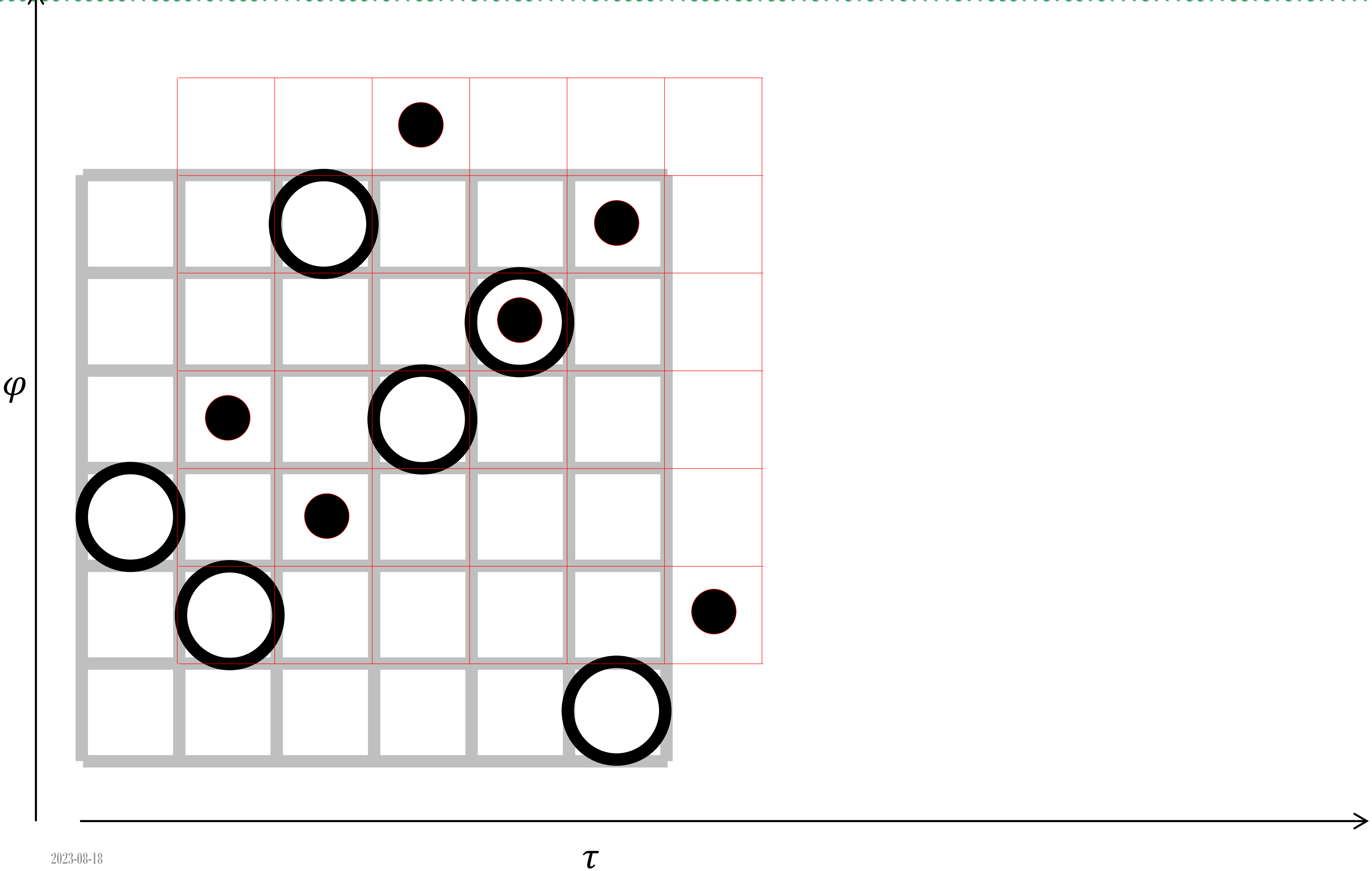
Autocorrelation at $(\tau=1, \varphi=0)$

1000000100000011000010100011110010001011001110101001111101000011100010010011011010110111101100011010010111011100110010101011111



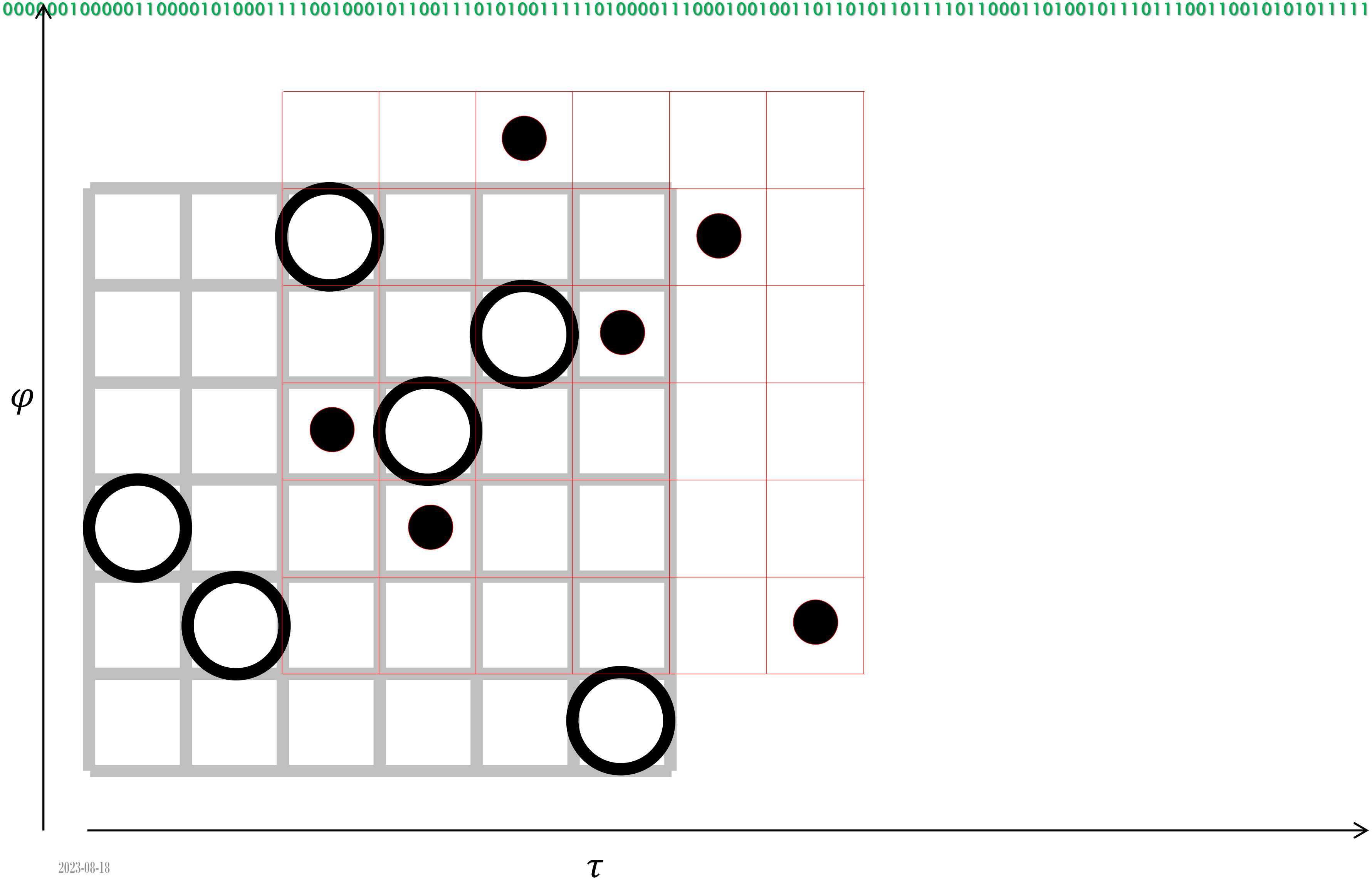
Autocorrelation at (τ=1, φ=1)

1000000100000011000010100011110010001011001110101001111101000011100010010011011010110111101100011010010111011100110010101011111111



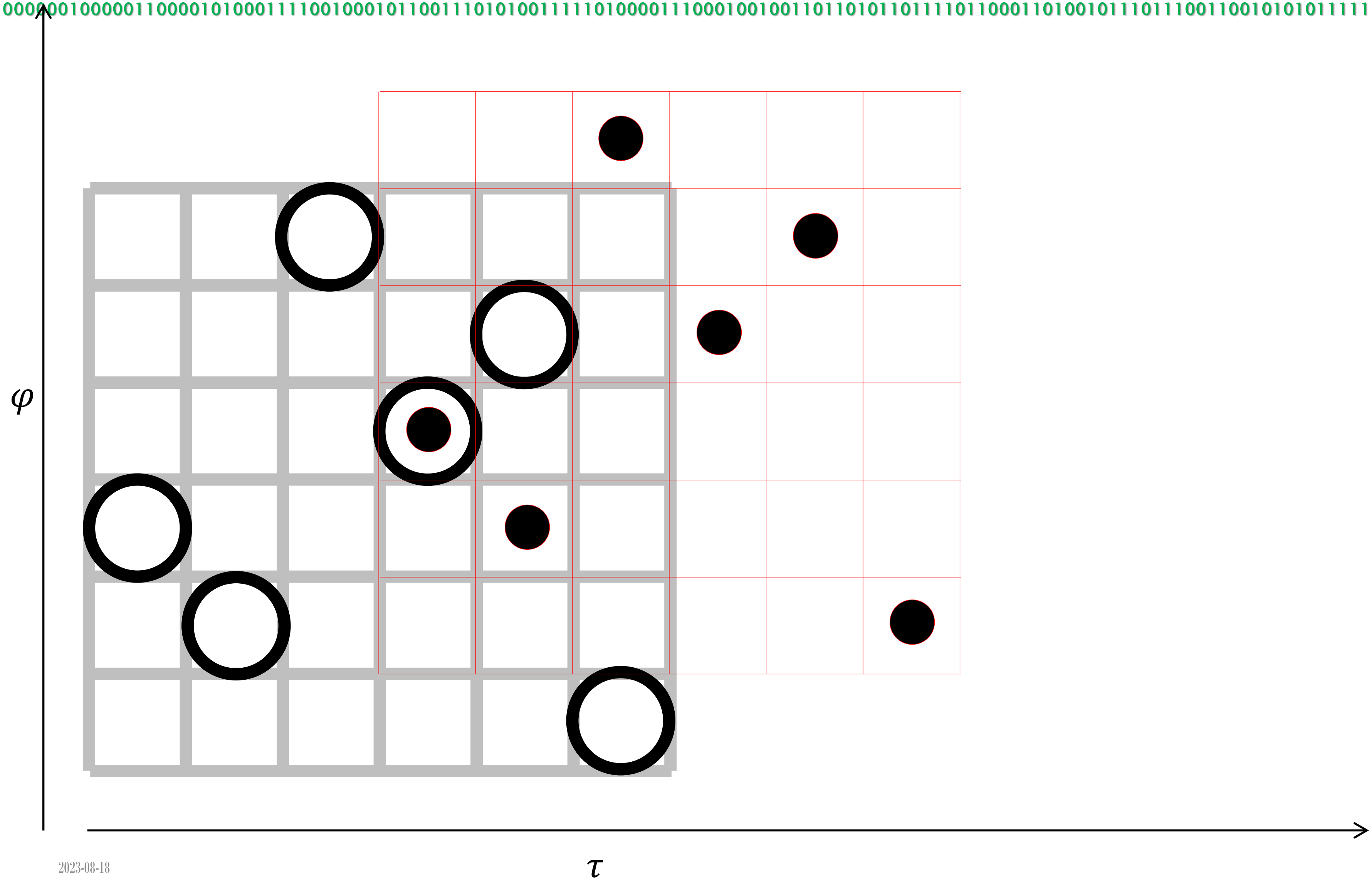
Autocorrelation at (τ=2, φ=1)

1000000100000011000010100011110010001011001110101001111101000011100010010011011010110111101100011010010111011100110010101011111



Autocorrelation at (τ=3, φ=1)

1000000100000011000010100011110010001011001110101001111101000001110001001001101101011011110110001101001011101110011001010101111111



discrete version of ambiguity function

Full (non-periodic) Autocorrelation Function

1000000100000110000101000

01110111001100101010111111

φ

		1								
				1		1				
			1				1	1		
1				1			1		1	
	1		1	1		1		1		
					6					
		1		1		1	1		1	
	1		1			1				1
		1	1				1			
				1		1				
								1		

$\tau = 0, \varphi = 0$

τ

Main Contribution of this talk

- Definition of **ZCZ sonar sequences** and **ZCZ-DD sonar sequences**
- **Theorem 1** on the upper bound on r for (m, n, r) ZCZ Sonar sequences
- **Theorem 2** on the construction of (m, n, r) ZCZ Sonar sequences for $r \geq 3$ with $m = r^2 - 1$ and $n > 1$
  - **Theorem 5 (corollary)** on the range of r for $(m, \boldsymbol{n} = \boldsymbol{m}, r)$ **ZCZ Sonar sequences**
- **(new) Definition** of **optimal** (m, n, r) ZCZ sonar sequence
- **Theorem 3** on the construction of $(q-4, q-4,\ \boldsymbol{r} = \mathbf{2})$ **ZCZ-DD sonar sequence** from the Lempel construction of Costas arrays
- **Theorem 4** on the construction of $(p, p-1,\ \boldsymbol{r} = \mathbf{2})$ **ZCZ-DD sonar sequence** from the Welch construction of Costas arrays

ZCZ sonar sequences

Definition (ZCZ sonar sequences)

An (m, n, r) **ZCZ sonar sequence** is a function $f: \{1, 2, \dots, n\} \to \{1, 2, \dots, m\}$

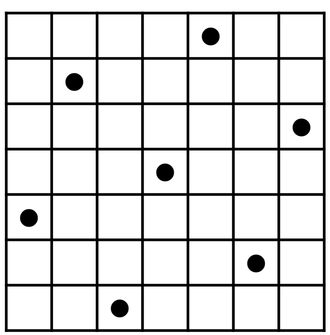
such that its autocorrelation satisfies

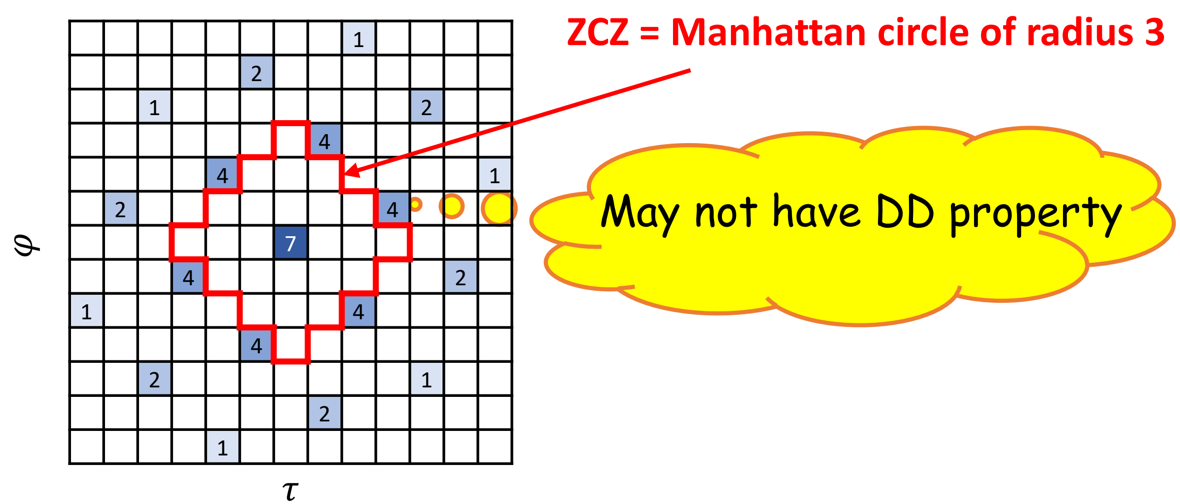
$$C(\varphi, \tau) = 0 \text{ for all } (\tau, \varphi) \text{ with } |\tau| + |\varphi| \le r \text{ except for } (0, 0).$$

where r is the zone radius in the **Manhattan metric**.

Example

A $(7, 7, r = 3)$ ZCZ sonar array and its autocorrelation.





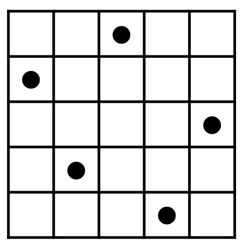
[13] E. F. Kraus, “Taxicab Geometry: An Adventure in Non-Euclidean Geometry,” New York, USA: Dover Publications, 1986.

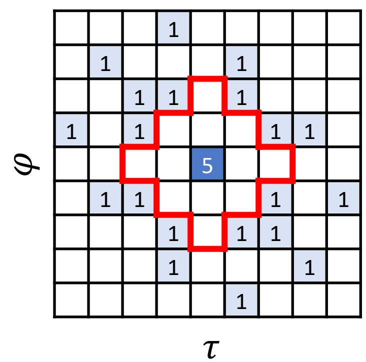
ZCZ-DD sonar sequences

Definition (ZCZ-DD sonar sequences)

An (m, n, r) **ZCZ-DD sonar sequence** is an (m, n, r) ZCZ sonar sequence with **DD property**.

Example A $(5, 5, 2)$ ZCZ-DD sonar array and its autocorrelation.

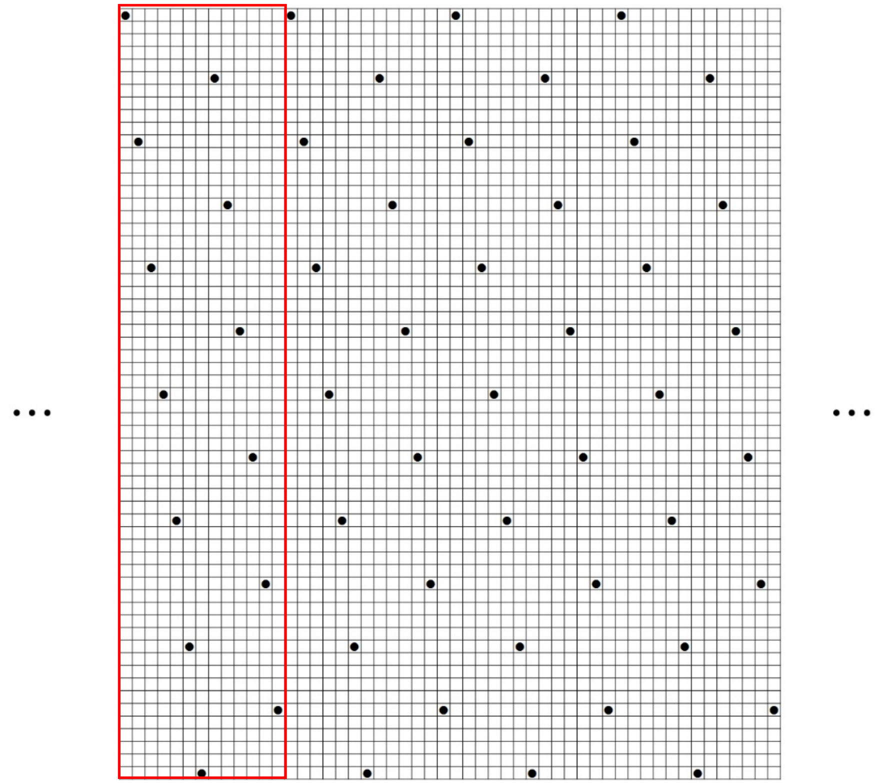




DD property guarantees that all out-of-phase values are at most 1

in addition to ZCZ property

- A ZCZ-DD sonar sequence is a sonar sequence.
- A ZCZ-DD sonar sequence is always a ZCZ sonar sequence, but not conversely.
- A ZCZ sonar sequence may not have DD property.



(61,52,10) ZCZ Sonar sequence
by computer search

- It has a **periodic structure** of a period of 13 columns repeating 4 times.
- Essentially, it gives a family of **$(61, n, 10)$ ZCZ sonar sequences** for any positive integer n.
- **(conventional) Optimal ZCZ sequence** cannot be determined !!!

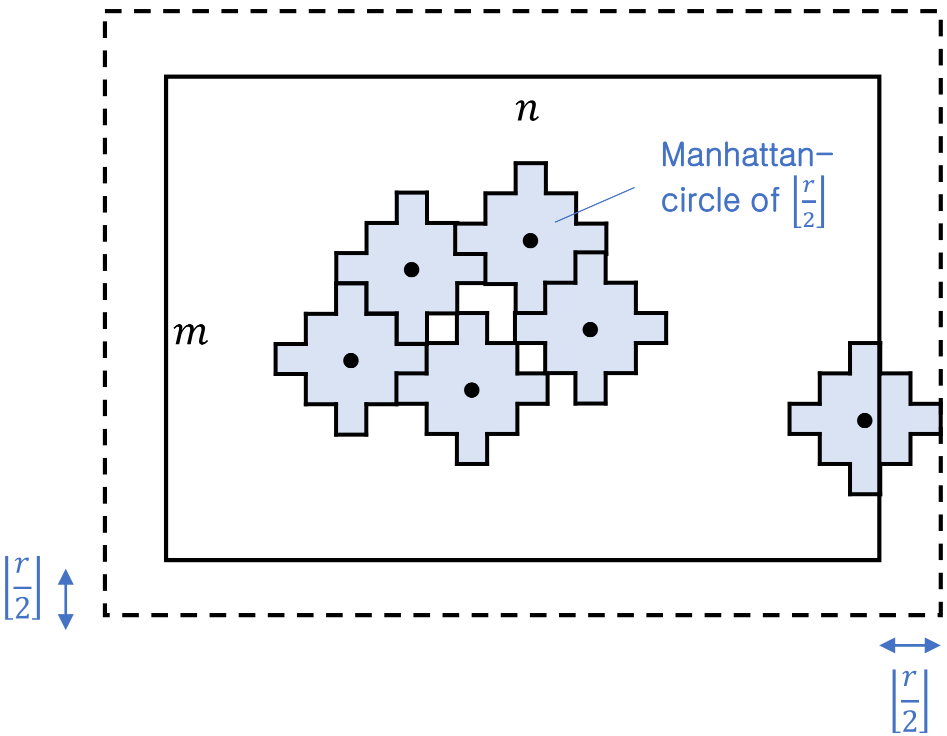
Theorem 1

Theorem 1

For an (m, n, r) ZCZ sonar sequence with $n \geq m \geq 3$,

$$r \leq \left\lfloor \frac{m + \sqrt{m^2 + 2n(n-2)(m-1)}}{n-2} \right\rfloor + 1.$$

Analogous to Hamming bound in Coding Theory.



The size of Manhattan-circle of radius $\left\lfloor \frac{r}{2} \right\rfloor$:

$$\left| s\left(\left\lfloor \frac{r}{2} \right\rfloor\right) \right| = 1 + 2\left\lfloor \frac{r}{2} \right\rfloor + 2\left\lfloor \frac{r}{2} \right\rfloor^2.$$

n times this area cannot be larger than the area given by

$n + \left\lfloor \frac{r}{2} \right\rfloor$ times $m + \left\lfloor \frac{r}{2} \right\rfloor$

Some upper bounds on r from Theorem 1

Upper bound		n																	
		3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
	3	**8** $_{1}$	5	4	4	4	3	3	3	3	3	3	3	3	3	3	3	3	3
	4		**7** $_{2}$	5	5	4	4	4	4	4	4	4	4	3	3	3	3	3	3
	5			**6** $_{2}$	5	5	5	5	4	4	4	4	4	4	4	4	4	4	4
	6				**6** $_{2}$	6	5	5	5	5	5	5	4	4	4	4	4	4	4
	7					**6** $_{3}$	6	6	5	5	5	5	5	5	5	5	5	5	5
	8						**6** $_{3}$	6	6	6	5	5	5	5	5	5	5	5	5
	9							**7** $_{3}$	6	6	6	6	6	6	5	5	5	5	5
m	10								**7** $_{3}$	6	6	6	6	6	6	6	6	6	6
	11									**7** $_{3}$	7	6	6	6	6	6	6	6	6
	12										**7** $_{4}$	7	7	7	6	6	6	6	6
	13											**7** $_{4}$	7	7	7	7	7	7	6
	14												**7** $_{4}$	7	7	7	7	7	7
	15													**7** $_{4}$	7	7	7	7	7
	16														**8** $_{4}$	7	7	7	7

these are true values found by computer

A construction of (m, n, r) ZCZ sonar sequence

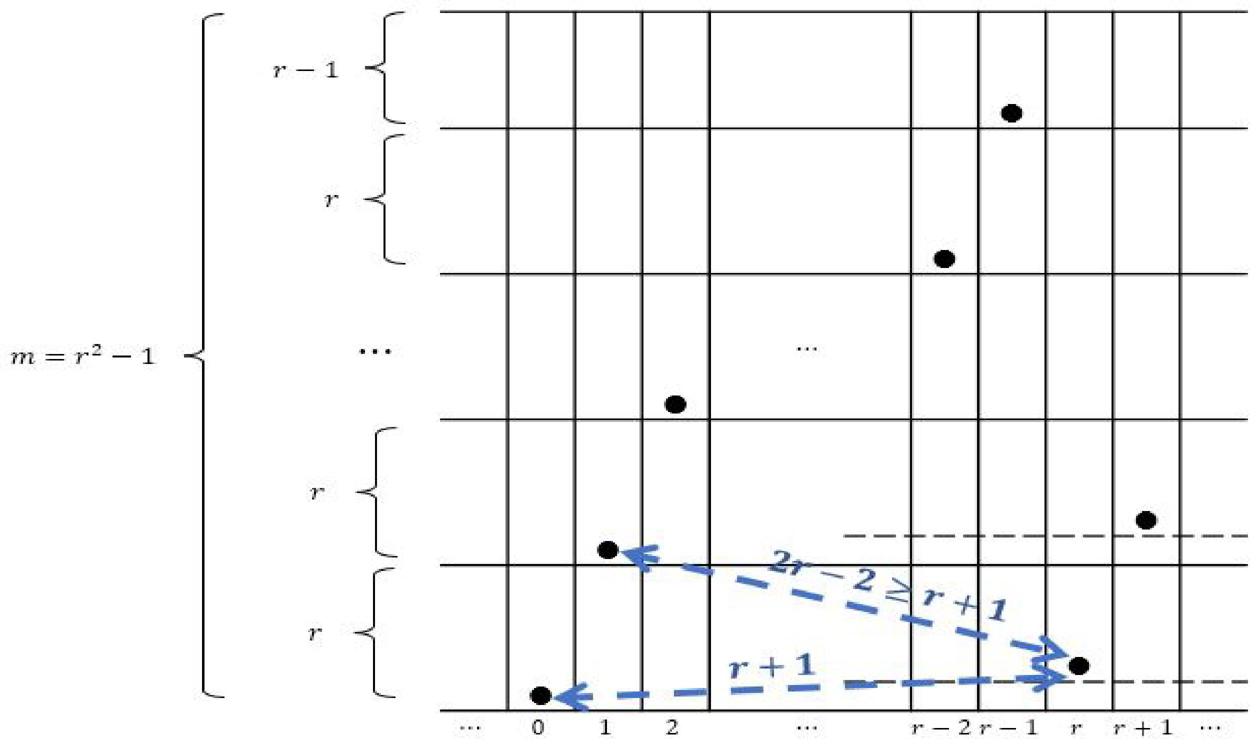
Theorem 2

For **any** positive integers n and $r \geq 3$ with $m = r^2 - 1$,

the function $f: \{0, 1, \dots, n-1\} \rightarrow \{0, 1, \dots, m-1\}$ defined by

$$f(j) = rj \pmod{m}$$

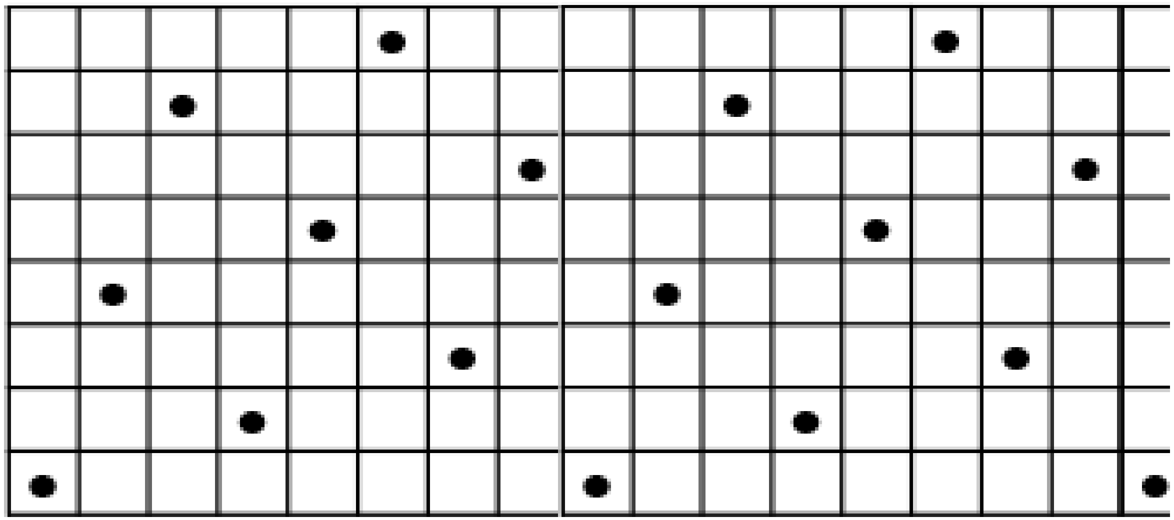
is an (m, n, r) ZCZ sonar sequence



Example

An $(8, n, 3)$ ZCZ sonar array.

…

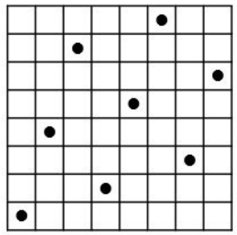


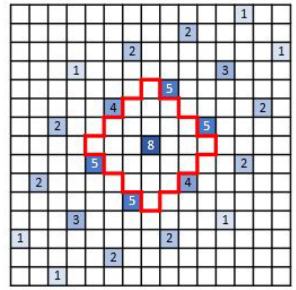
…

- Meaningless to talk about conventional '**optimal**' $(8, n, 3)$ ZCZ sonar sequence of the largest value n
- We may define an **optimal** (m,n,r) ZCZ sonar sequence with the largest value of r for given m and n.

3 examples

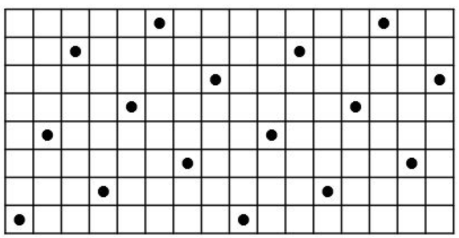
(8,8,3) ZCZ sonar array

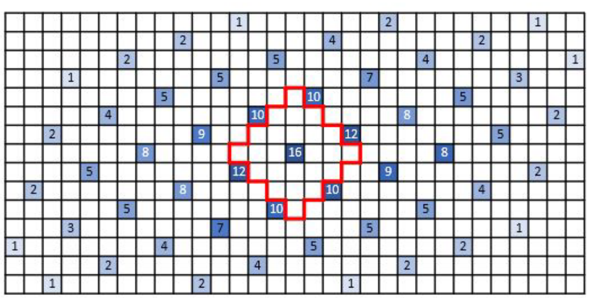




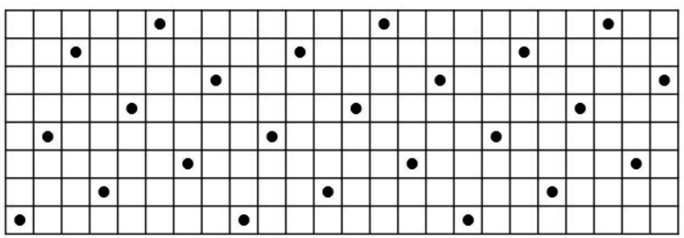
optimal (8,8,3) ZCZ sonar sequence
in the sense that 3 is the largest for given n=m=8

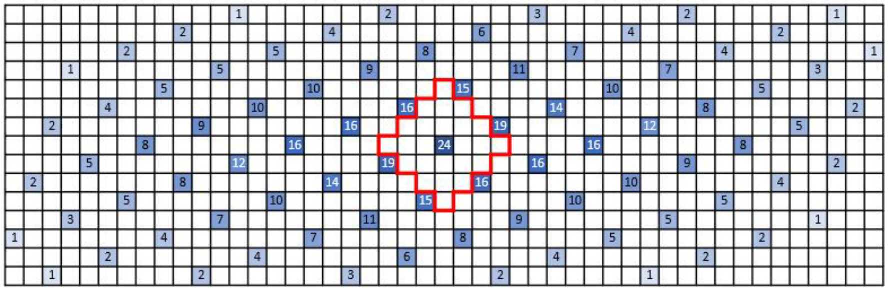
(8,16,3) ZCZ sonar array





(8,24,3) ZCZ sonar array





An interesting sub-case: $n = m$.

The upper bound in Theorem 1 can be further simplified to

$$r < \left\lfloor \frac{m(1+\sqrt{2m})}{m-2} \right\rfloor + 1 \approx \lfloor 2 + \sqrt{2m} \rfloor.$$

By construction, Theorem 2 for $n = m$ gives a lower bound.

Theorem 5

For an (m, m, r) ZCZ sonar sequence with $m \geq r \geq 3$, we have

$$\lceil \sqrt{m+2} \rceil \leq r < \lfloor 2 + \sqrt{2m} \rfloor.$$

- Roughly, it says

$$\sqrt{m} \lesssim r \lesssim \sqrt{2m}.$$

Exhaustive search for the true maximum r in $(m, n = m, r)$ ZCZ sonar sequences

m	true max	upper bound	m	true max	upper bound	m	true max	upper bound
3	1	8	23	5	9	43	8	11
4	2	7	24	5	9	44	8	11
5	2	6	25	6	9	45	8	11
6	2	6	26	6	9	46	8	11
7	3	6	27	6	9	47	8	11
8	3	7	28	6	9	48	8	12
9	3	7	29	6	9	49	9	12
10	3	7	30	6	10	50	9	12
11	3	7	31	7	10	51	9	12
12	4	7	32	7	10	52	9	12
13	4	7	33	7	10	53	9	12
14	4	7	34	7	10	54	9	12
15	4	7	35	7	10	55	9	12
16	4	8	36	7	10	56	9	12
17	5	8	37	7	10	57	9	12
18	5	8	38	7	10	58	9	12
19	5	8	39	7	11	59	9	13
20	5	8	40	8	11	60	10	13
21	5	8	41	8	11	61	10	13
22	5	8	42	8	11	62	10	13

Does not exist a $(8, 8, 4)$ ZCZ sonar sequence.

quite GOOD
as m increases!

Some open problems

For (m,n,r) ZCZ sonar sequences

1) Improve the upper bound on r for (m,n,r) **ZCZ sonar sequences** in Theorem 1.

For (m,n,r) ZCZ-DD sonar sequences

2) Find the upper bound on r for given m and n.
3) Find the (largest) value n for given r and m.
4) Find some systematic constructions for the best (large) values of $r > 2$.

THE END