Some New Binary Locally Repairable Codes with Availability based on Golomb Rulers

Hyojeong Choi and Hong-Yeop Song
School of Electrical and Electronic Engineering, Yonsei University
Seoul, Korea

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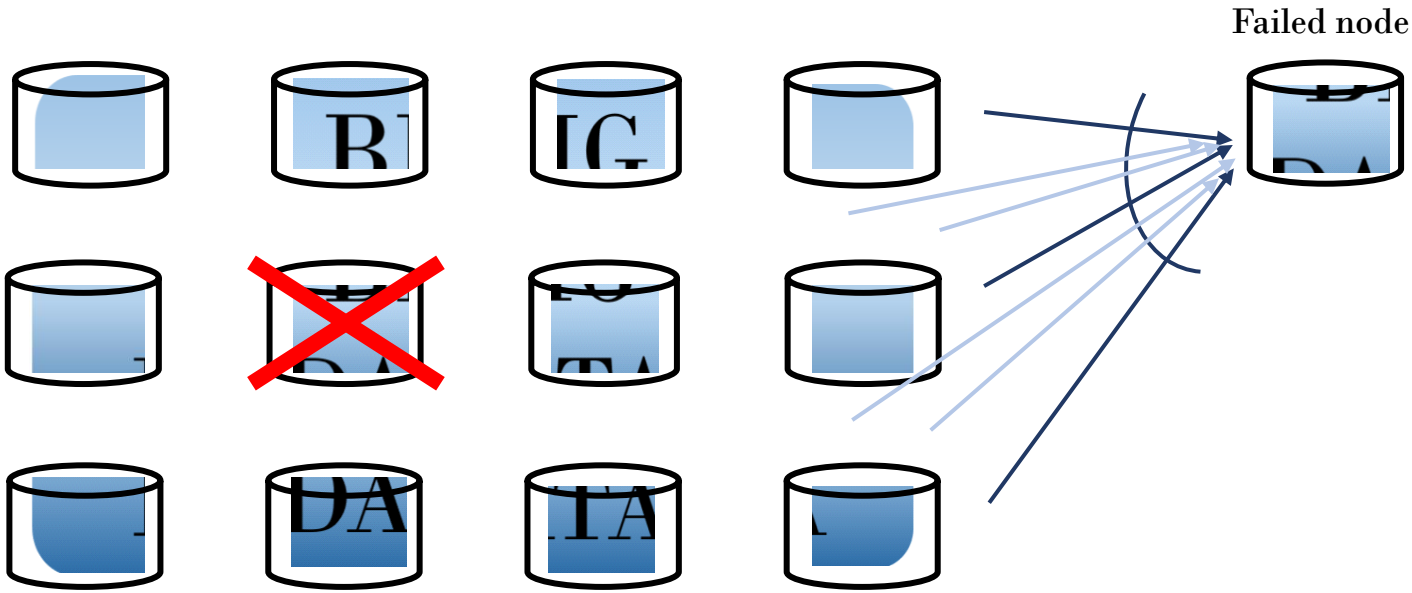
ICTC

Contents

Locally Repairable Code

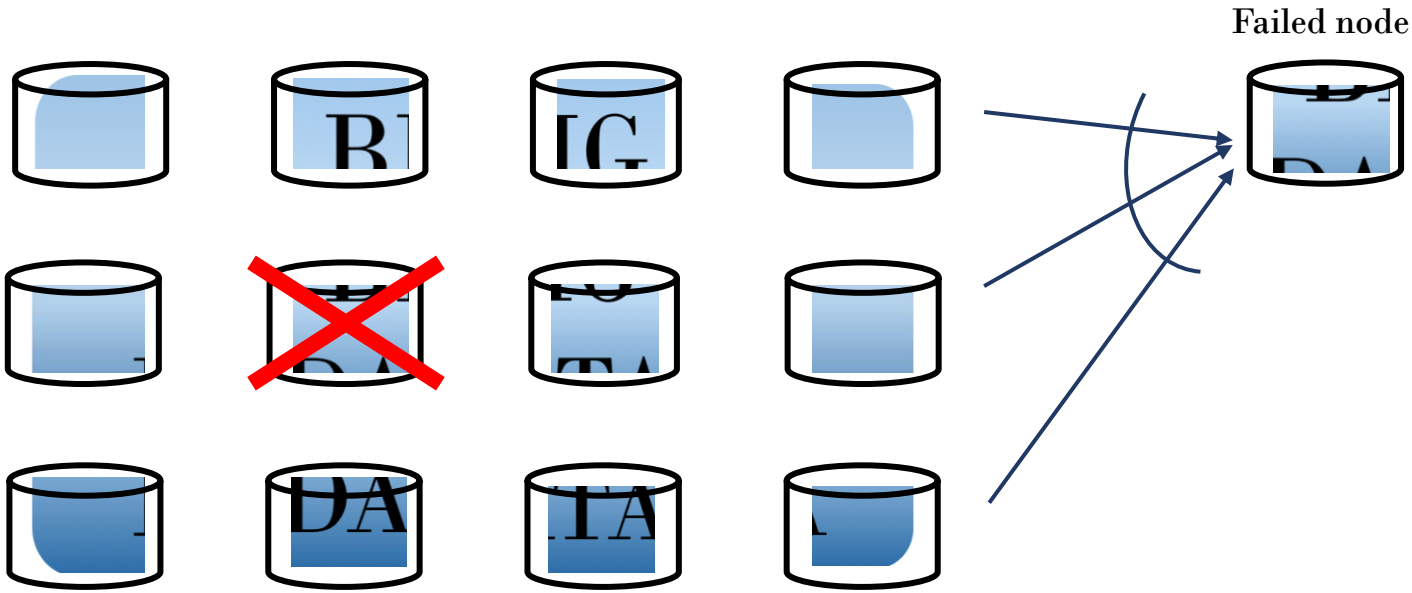
- To guarantee the reliability against node failures, various coding techniques have been applied



Locally Repairable Code

[Gopalan et al. 12]

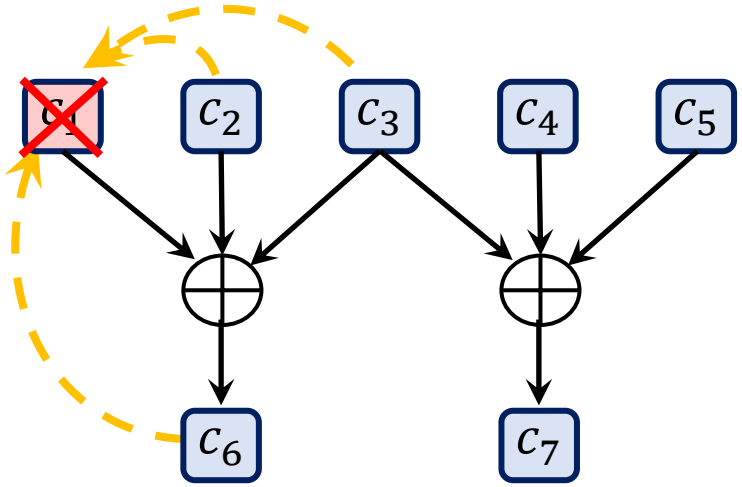
- Locally repairable code (LRC) just needs a small number of nodes to repair the single node failure



Locally Repairable Code

- **Locality**

  The number of nodes accessed to repair a single node failure



Locality of $c_1 \Rightarrow 3$

- Code C has locality r:

  All coded symbols have the locality at most r.

  C is denoted as $[n,k,r]$ LRC.

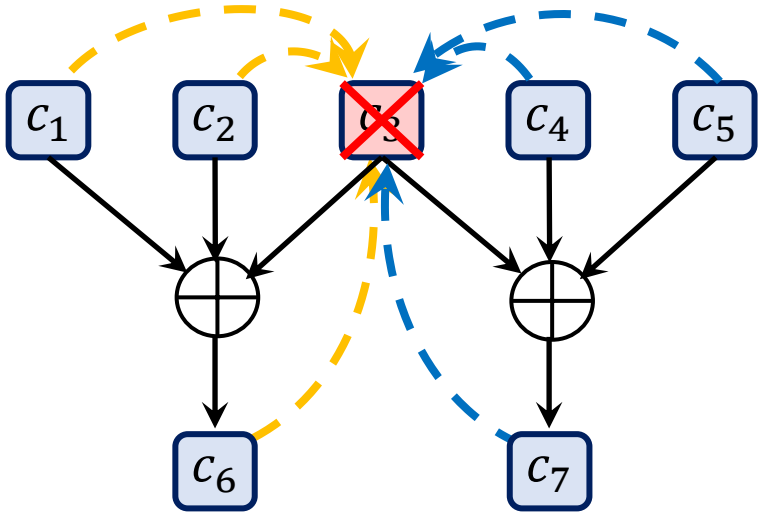
Locally Repairable Code

- **Availability**

  The number of disjoint repair sets to repair a single node failure

Repair set of $c_3 \Rightarrow \{c_1, c_2, c_6\}$

Repair set of $c_3 \Rightarrow \{c_4, c_5, c_7\}$



Locality of c_3 $\Rightarrow 3$

Availability of c_3 $\Rightarrow 2$

LRCs for multiple erasures

- ***t*-parallel-recovery LRCs**
  - The repaired erasure **cannot** participate in the repair process of the unrepaired erasures

E.g. Erasures: $1^{st}, 2^{nd}$ symbol

Repaired locally and parallelly

i^{th} symbol	Repair set
1	{2,3} and {**4**, **7**}
2	{1,3} and {**5**, **8**}

- ***u*-sequential-recovery (u-seq) LRCs**
  - The repaired erasure **can** participate in the repair process of the unrepaired erasures

E.g. Erasures: $1^{st}, 2^{nd}, 7^{th}$ symbol

Locally repaired by order $7 \rightarrow 2 \rightarrow 1$

$2 \rightarrow 7 \rightarrow 1$

$2\ and\ 7 \rightarrow 1$

i^{th} symbol	Repair set
1	{**2**, **3**} and {**4**, **7**}
2	{1,3} and {**5**, **8**}
7	{1,4} and {**8**, **9**}

❖ **The *repair time* is defined the maximum number of steps required to repair any *u* erasures.**

Connections between LRCs for multiple erasures and regular LDPC

[8] Z. Jing and H. -Y. Song, "Girth-Based Sequential-Recovery LRCs," *IEEE Access*, vol. 10, pp. 126156-126160, 2022.

Known Fact 1 [8].

1) A linear block code is a u-seq LRC with locality r if its parity check matrix satisfies the following:
   (i) the **girth** is $2(u+1)$,
   (ii) the **column weight** is at least 2, and
   (iii) the **row weight** is at most $r+1$.

2) The **repair time** of u-seq LRC defined above is at most $\lceil u/2 \rceil$.

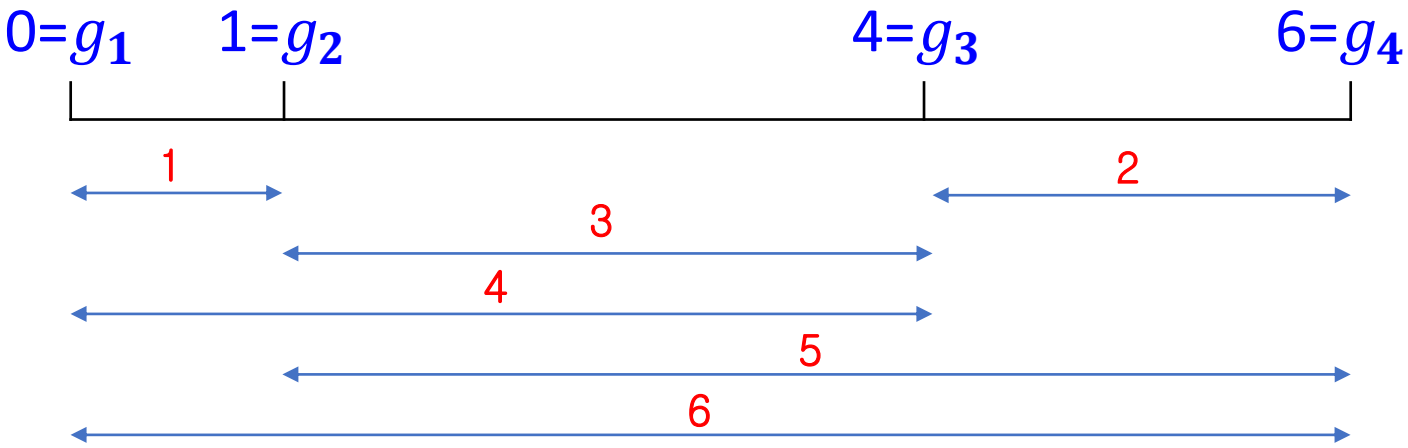
Golomb rulers

s-mark Golomb ruler:

A set of s integers $0 = g_1 < g_2 < \ \ldots \ < g_s$ such that $g_j - g_i$ for $i < j$ are all distinct.

Example: 4-mark Golomb ruler

$$\{ g_1, g_2, g_3, g_4 \} = \{ 0, 1, 4, 6 \}$$



Interval **distances** are all distinct

Constructing a parity check matrix for QC-LDPC codes

exponent matrix $E = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 4 & 6 \end{pmatrix}$ over the integers and a positive integer m

parity check matrix $H = \begin{bmatrix} I^{(1)} & I^{(1)} & I^{(1)} & I^{(1)} \\ I^{(0)} & I^{(1)} & I^{(4)} & I^{(6)} \end{bmatrix}$ over GF(2) of size $2m \times 4m$

where $I^{(j)}$ is the $m \times m$ identity matrix whose columns are circularly shifted j times

Example: **Select $m = 13$** $\Rightarrow$ H becomes a 26×52 matrix with

$I^{(0)} = I =$

1	0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0	0	0
0	0	0	0	0	0	1	0	0	0	0	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	0	0	0	1

$I^{(1)} =$

0	1	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0	0	0
0	0	0	0	0	0	1	0	0	0	0	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	0	0	0	1
1	0	0	0	0	0	0	0	0	0	0	0	0

$I^{(4)} =$

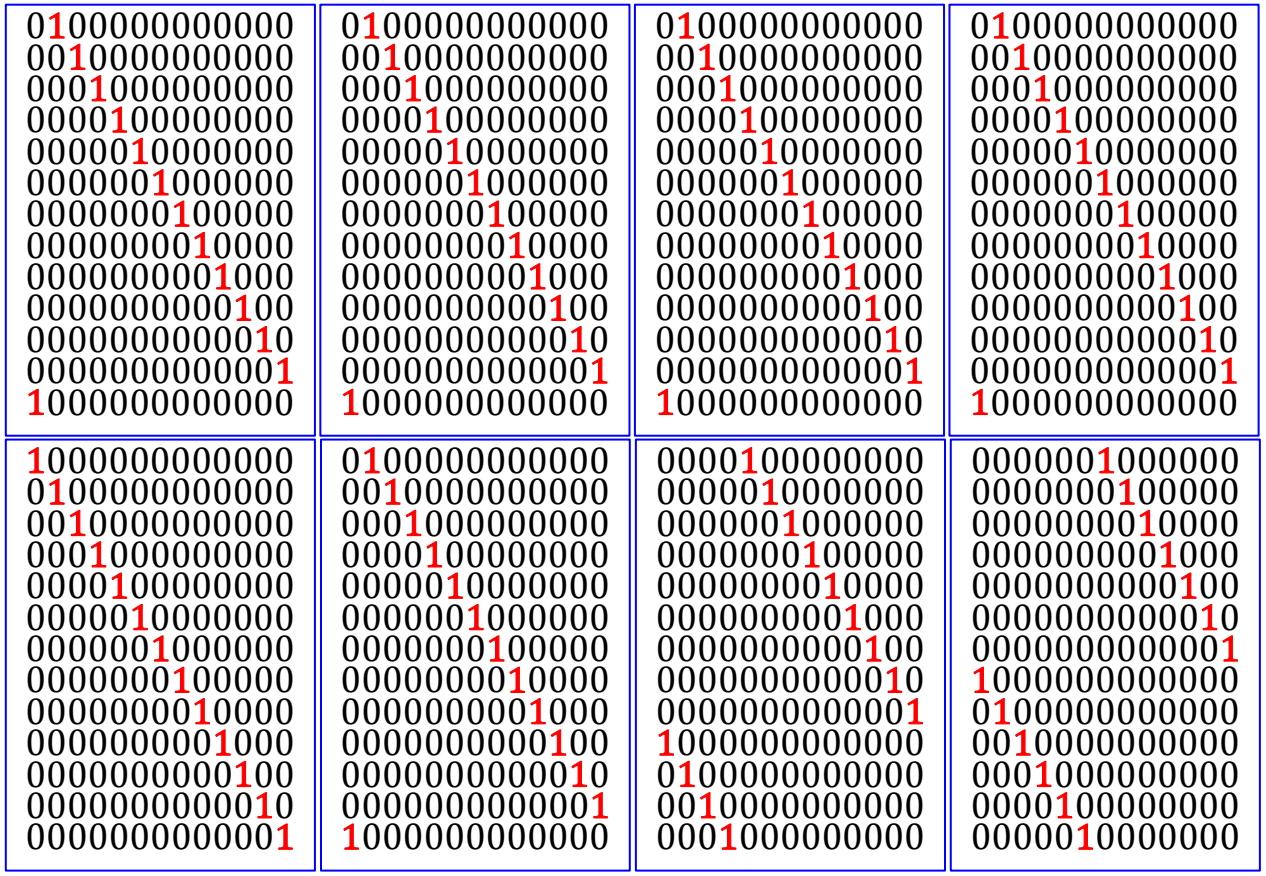
0	0	0	0	1	0	0	0	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0	0	0
0	0	0	0	0	0	1	0	0	0	0	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	0	0	0	1
1	0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0

$I^{(6)} =$

0	0	0	0	0	0	1	0	0	0	0	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	0	0	0	1
1	0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0	0	0

- **[5] M.P.C. Fossorier, *IEEE Trans. Inf. Theory*, 2004** gives a necessary condition in E for the existence of even-cycles in H

$$E = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 4 & 6 \end{pmatrix} \Rightarrow H = \begin{bmatrix} I^{(1)} & I^{(1)} & I^{(1)} & I^{(1)} \\ I^{(0)} & I^{(1)} & I^{(4)} & I^{(6)} \end{bmatrix}$$



We will see later that **this parity check matrix** gives

a 5-seq LRC with parameters: length $n = 52$, dimension $k = 25$, locality $r = 3$ and availability $t = 2$

1. Preliminary

1) Locally repairable code (LRC)

2) LRCs for multiple erasure

3) scheme for generating a parity check matrix for QC-LDPC codes

2. 5-seq LRCs based on Golomb Rulers

3. Concluding Remark

Main Result

Construction of 5-seq LRCs with availability $t = 2$ from a Golomb Ruler

- We need an s-mark Golomb ruler: $0 = g_1 < g_2 < \ldots < g_s$ of length g_s.
  - distances are all distinct ⇒ $g_i \neq g_j$ for $i < j$

    as well as $g_j - g_i \neq g_l - g_k$ for $i < j$ and $k < l$ with $i \neq k$

- We need to choose a positive integer m satisfying the following three conditions:
  1) $g_i \neq g_j \pmod m$ for $i < j$
  2) $g_j - g_i \neq -(g_l - g_k) \pmod m$ for $i < j$ and $k < l$ but not necessarily $i \neq k$
  3) GCD of m and all the distances $(g_j - g_i)'s$ is collectively 1 (not pairwise)

$\Leftarrow m > g_s$

$\Leftarrow m > 2g_s$ and some more

$\Leftarrow$ any m when $g_j - g_i = 1$ for some $i < j$

Main Result

Construction of 5-seq LRCs with availability t=2 from a Golomb Ruler

Theorem 1.

- Let $0 = g_1 < g_2 < \ldots < g_s$ of length g_s be an s-mark Golomb ruler.
- Let m be a positive integer satisfying the three conditions above.
- Let the exponent matrix E be of size $2 \times s$ in which the first row is a constant number c and the second row is $g_1, g_2, \ldots, g_s$.
- Construct a binary $2m \times sm$ matrix H by substituting $I^{(c)}$ or $I^{(g_j)}$ into the positions of E.

Then, H is a parity check matrix of a **5-seq LRC** with

$n = ms, (s-2)m + 1 \leq k \leq (s-2)m + \left\lfloor \frac{s}{m} \right\rfloor$, $r = s - 1$ and availability $t = 2$.

Example

- 4-mark Golomb ruler: $\{0,1,4,6\}$
- Choose $m = 13 > 6 = g_s$ and $13 > 12 = 2g_s$
  - ✓ One of the distances is 1. So the third condition is automatically satisfied.

$$E = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 4 & 6 \end{pmatrix} \Rightarrow H = \begin{bmatrix} I^{(1)} & I^{(1)} & I^{(1)} & I^{(1)} \\ I^{(0)} & I^{(1)} & I^{(4)} & I^{(6)} \end{bmatrix}$$

0100000000000	0100000000000	0100000000000	0100000000000
0010000000000	0010000000000	0010000000000	0010000000000
0001000000000	0001000000000	0001000000000	0001000000000
0000100000000	0000100000000	0000100000000	0000100000000
0000010000000	0000010000000	0000010000000	0000010000000
0000001000000	0000001000000	0000001000000	0000001000000
0000000100000	0000000100000	0000000100000	0000000100000
0000000010000	0000000010000	0000000010000	0000000010000
0000000001000	0000000001000	0000000001000	0000000001000
0000000000100	0000000000100	0000000000100	0000000000100
0000000000010	0000000000010	0000000000010	0000000000010
0000000000001	0000000000001	0000000000001	0000000000001
1000000000000	1000000000000	1000000000000	1000000000000
1000000000000	0100000000000	0000100000000	0000001000000
0100000000000	0010000000000	0000010000000	0000000100000
0010000000000	0001000000000	0000001000000	0000000010000
0001000000000	0000100000000	0000000100000	0000000001000
0000100000000	0000010000000	0000000010000	0000000000100
0000010000000	0000001000000	0000000001000	0000000000010
0000001000000	0000000100000	0000000000100	0000000000001
0000000100000	0000000010000	0000000000010	1000000000000
0000000010000	0000000001000	0000000000001	0100000000000
0000000001000	0000000000100	1000000000000	0010000000000
0000000000100	0000000000010	0100000000000	0001000000000
0000000000010	0000000000001	0010000000000	0000100000000
0000000000001	1000000000000	0001000000000	0000010000000

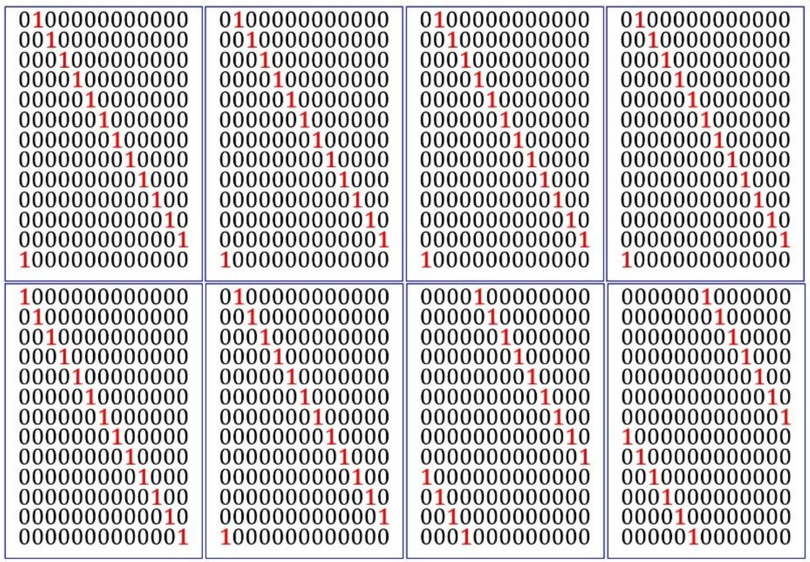
This parity check matrix
gives

a 5-seq LRC
with parameters:
length $n = 52$,
dimension $k = 25$,
locality $r = 3$
and availability $t = 2$

Some Remarks

This constant can be any number.

$$E = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 4 & 6 \end{pmatrix} \Rightarrow H = \begin{bmatrix} I^{(1)} & I^{(1)} & I^{(1)} & I^{(1)} \\ I^{(0)} & I^{(1)} & I^{(4)} & I^{(6)} \end{bmatrix}$$



Some Remarks

This can be any Golomb ruler. The number of marks determines its size.

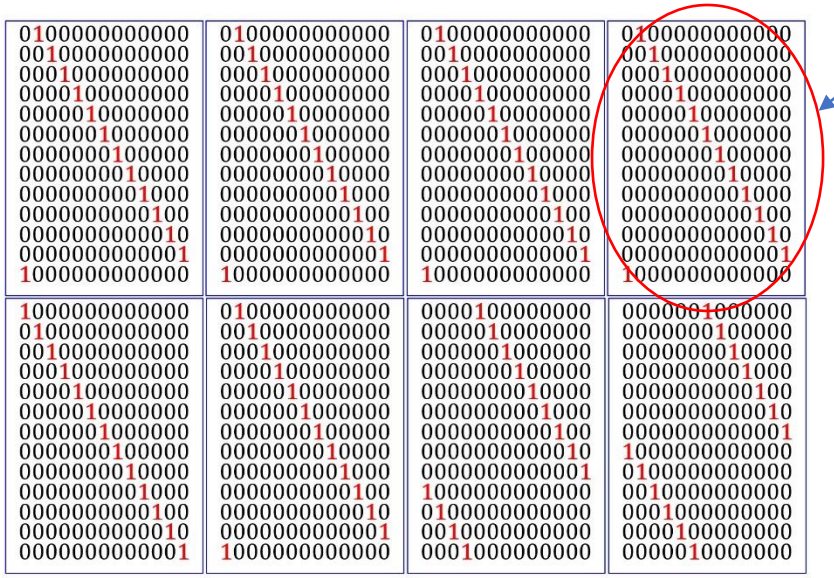
$$E = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 4 & 6 \end{pmatrix} \Rightarrow H = \begin{bmatrix} I^{(1)} & I^{(1)} & I^{(1)} & I^{(1)} \\ I^{(0)} & I^{(1)} & I^{(4)} & I^{(6)} \end{bmatrix}$$

0100000000000	0100000000000	0100000000000	0100000000000
0010000000000	0010000000000	0010000000000	0010000000000
0001000000000	0001000000000	0001000000000	0001000000000
0000100000000	0000100000000	0000100000000	0000100000000
0000010000000	0000010000000	0000010000000	0000010000000
0000001000000	0000001000000	0000001000000	0000001000000
0000000100000	0000000100000	0000000100000	0000000100000
0000000010000	0000000010000	0000000010000	0000000010000
0000000001000	0000000001000	0000000001000	0000000001000
0000000000100	0000000000100	0000000000100	0000000000100
0000000000010	0000000000010	0000000000010	0000000000010
0000000000001	0000000000001	0000000000001	0000000000001
1000000000000	1000000000000	1000000000000	1000000000000

1000000000000	0100000000000	0000100000000	0000001000000
0100000000000	0010000000000	0000010000000	0000000100000
0010000000000	0001000000000	0000001000000	0000000010000
0001000000000	0000100000000	0000000100000	0000000001000
0000100000000	0000010000000	0000000010000	0000000000100
0000010000000	0000001000000	0000000001000	0000000000010
0000001000000	0000000100000	0000000000100	0000000000001
0000000100000	0000000010000	0000000000010	1000000000000
0000000010000	0000000001000	0000000000001	0100000000000
0000000001000	0000000000100	1000000000000	0010000000000
0000000000100	0000000000010	0100000000000	0001000000000
0000000000010	0000000000001	0010000000000	0000100000000
0000000000001	1000000000000	0001000000000	0000010000000

Some Remarks

$$E = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 4 & 6 \end{pmatrix} \Rightarrow H = \begin{bmatrix} I^{(1)} & I^{(1)} & I^{(1)} & I^{(1)} \\ I^{(0)} & I^{(1)} & I^{(4)} & I^{(6)} \end{bmatrix}$$



The size m must be carefully selected for a given s-mark Golomb ruler.

- Easy choice is $m > 2g_s$
- There are possible values of m in the range of $g_s < m < 2g_s$
- We found the 5-seq LRC becomes **optimal** in the sense of...

Some Remarks

[1] S. B. Balaji, G. R. Kini, and P. V. Kumar, "A tight rate bound and a matching construction for locally recoverable codes with sequential recovery from any number of multiple erasures," in Proc. IEEE Int. Symp. Inf. Theory (ISIT), pp. 1778–1782, Jun. 2017.

- In [1], it was proposed that for $r \geq 3$, when u is odd and $\sigma = \left\lfloor \frac{u-1}{2} \right\rfloor$, the bound is as follows :

$$\frac{k}{n} \leq \frac{r^{\sigma+1}}{r^{\sigma+1} + 2\sum_{i=1}^{\sigma} r^i + (u-2\sigma)} \qquad (1)$$

TABLE I
EXAMPLES OF VARIOUS OPTIMAL 5-SEQ LRCS FROM **Theorem 1** USING OPTIMAL GOLOMB RULERS

s	optimal Golomb ruler	min m	n	k	code rate	rate bound (1) for $u = 5$	$2g_s + 1 \geq \min m$
4	$0, 1, 4, 6$	13	52	27	0.519	0.519	$13 = \min m$
5	$0, 2, 7, 8, 11$	21	105	64	0.610	0.610	$23 > \min m$
6	$0, 1, 4, 10, 12, 17$ $0, 1, 8, 11, 13, 17$	31	186	125	0.672	0.672	$35 > \min m$
7	$0, 2, 3, 10, 16, 21, 25$ $0, 2, 7, 13, 21, 22, 25$	49	343	246	0.717	0.717	$51 > \min m$
8	$0, 1, 4, 9, 15, 22, 32, 34$	69	552	415	0.752	0.752	$69 = \min m$
9	$0, 1, 5, 12, 25, 27, 35, 41, 44$	89	801	624	0.779	0.779	$89 = \min m$

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1) Locally repairable code (LRC)

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Concluding Remark

In this paper,

- We propose some new 5-seq LRCs with availability $t = 2$ based on Golomb rulers and some wellknown technique of generating the parity check matrix for the QC-LDPC codes.
- The proposed 5-seq LRCs can repair up to 5 erased symbols sequentially within 3 repair time.
- The code rate of the resulting codes using the optimal Golomb rulers in Table I turns out to be optimal.
- It is our current research to characterize for which Golomb rulers the resulting code becomes rate-optimal.

Thank you for listening