We are about to submit the manuscript of this presentation and many more into some journal. Therefore, please do not record this presentation, please!

Construction of 5-SEQ LRCs with Availability 2 from Golomb Rulers

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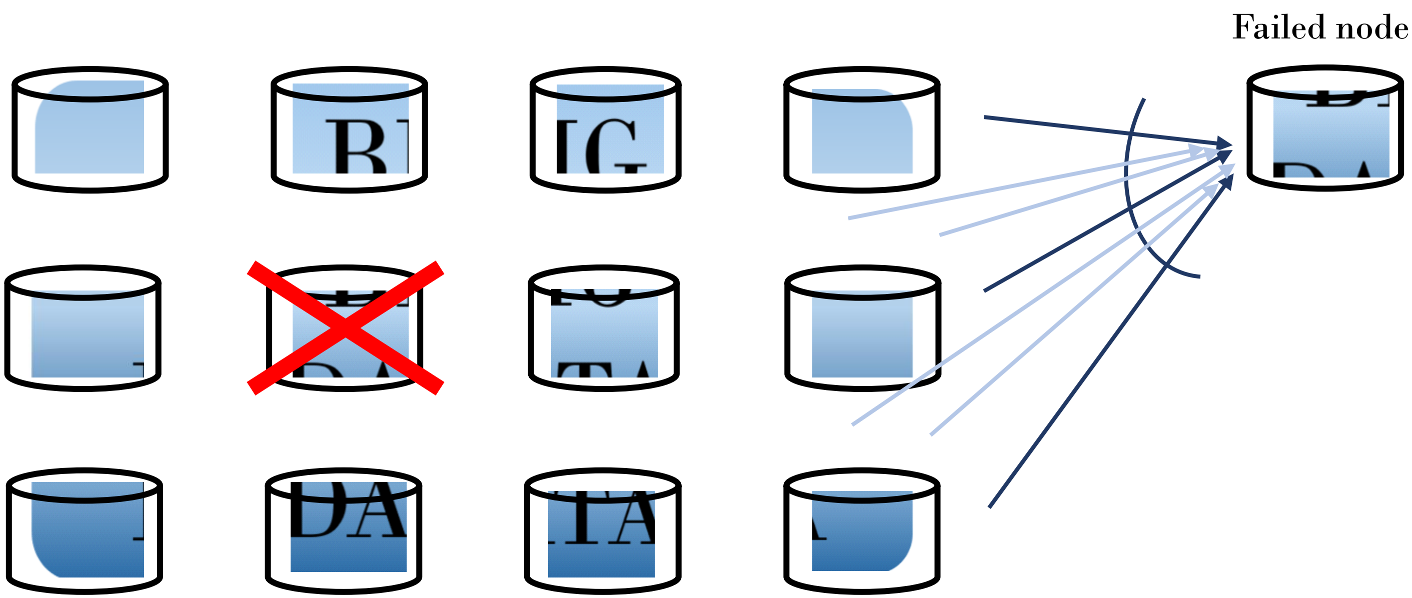
Contents

Locally Repairable Code

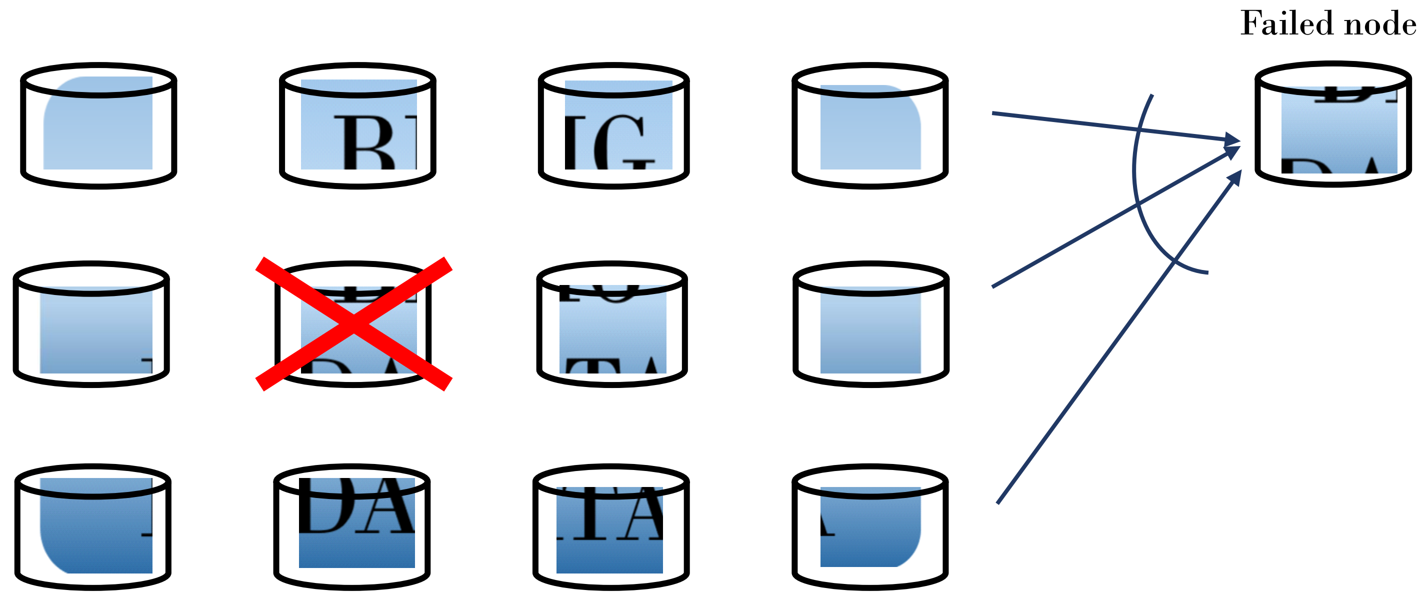
- To guarantee the reliability against node failures, various coding techniques have been applied



Locally Repairable Code

[Gopalan et al. 12]

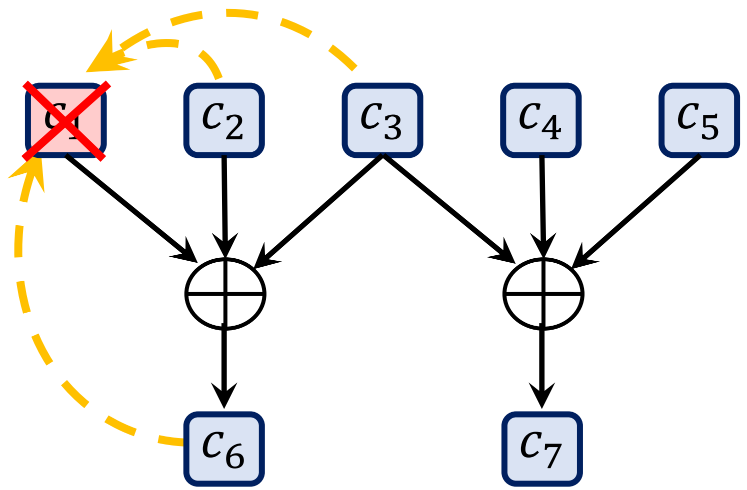
- Locally repairable code (LRC) just needs a small number of nodes to repair the single node failure



Locally Repairable Code

- **Locality**

  The number of nodes accessed to repair a single node failure



Locality of $c_1 \Rightarrow 3$

- Code C has locality r:

  All coded symbols have the locality at most r.

  C is denoted as $[n,k,r]$ LRC.

Locally Repairable Code

- **Availability**

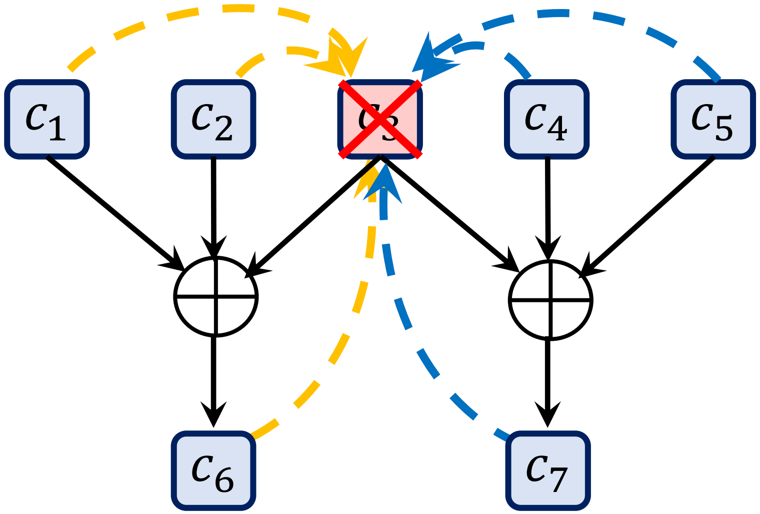
  The number of disjoint repair sets to repair a single node failure

Repair set of $c_3 \Rightarrow \{c_1, c_2, c_6\}$

Repair set of $c_3 \Rightarrow \{c_4, c_5, c_7\}$

Locality of $c_3 \Rightarrow 3$

Availability of $c_3 \Rightarrow 2$



LRCs for multiple erasures

- ***t*-parallel-recovery LRCs**
  - The repaired erasure **cannot** participate in the repair process of the unrepaired erasures

E.g. Erasures: $1^{st}, 2^{nd}$ symbol

Repaired locally and parallelly

i^{th} symbol	Repair set
1	{2,3} and $\{\mathbf{4}, \mathbf{7}\}$
2	{1,3} and $\{\mathbf{5}, \mathbf{8}\}$

- ***u*-sequential-recovery (u-seq) LRCs**
  - The repaired erasure **can** participate in the repair process of the unrepaired erasures

E.g. Erasures: $1^{st}, 2^{nd}, 7^{th}$ symbol

Locally repaired by order $7 \rightarrow 2 \rightarrow 1$

$2 \rightarrow 7 \rightarrow 1$

$2\ and\ 7 \rightarrow 1$

i^{th} symbol	Repair set
1	$\{\mathbf{2}, \mathbf{3}\}$ and $\{\mathbf{4}, \mathbf{7}\}$
2	{1,3} and $\{\mathbf{5}, \mathbf{8}\}$
7	{1,4} and $\{\mathbf{8}, \mathbf{9}\}$

❖ **The *repair time* is defined the maximum number of steps required to repair any u erasures.**

Connections between LRCs for multiple erasures and regular LDPC

[8] Z. Jing and H. -Y. Song, "Girth-Based Sequential-Recovery LRCs," *IEEE Access*, vol. 10, pp. 126156-126160, 2022.

Known Fact 1 [8].

1) A linear block code is a u-seq LRC with locality r if its parity check matrix satisfies the following:
   (i) the **girth** is $2(u + 1)$,
   (ii) the **column weight** is at least 2, and
   (iii) the **row weight** is at most $r + 1$.

2) The **repair time** of u-seq LRC defined above is at most $\lceil u/2 \rceil$.

Connections between LRCs for multiple erasures and regular LDPC

[8] Z. Jing and H. -Y. Song, "Girth-Based Sequential-Recovery LRCs," *IEEE Access*, vol. 10, pp. 126156-126160, 2022.

(**example**)

A binary **3**-seq LRC is constructed by the following parity-check matrix H with girth **8**.

$$H = \left(\begin{array}{ccc|ccc|ccc} 1&1&1&0&0&0&0&0&0\\ 0&0&0&1&1&1&0&0&0\\ 0&0&0&0&0&0&1&1&1\\ \hline 1&0&0&1&0&0&1&0&0\\ 0&1&0&0&1&0&0&1&0\\ 0&0&1&0&0&1&0&0&1 \end{array}\right)$$

disjoint repair group

Connections between LRCs for multiple erasures and regular LDPC

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(example)

A binary **3**-seq LRC is constructed by the following parity-check matrix H with girth **8**.

$$H = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

Erasures: $1^{st}, 4^{nd}, 7^{th}$ symbols

i^{th} symbol	Repair set
1	$\{\mathbf{2},\mathbf{3}\}$ and $\{4,7\}$
4	$\{\mathbf{5},\mathbf{6}\}$ and $\{1,7\}$
7	$\{\mathbf{8},\mathbf{9}\}$ and $\{1,4\}$

Connections between LRCs for multiple erasures and regular LDPC

[8] Z. Jing and H. -Y. Song, "Girth-Based Sequential-Recovery LRCs," *IEEE Access*, vol. 10, pp. 126156-126160, 2022.

(example)

A binary **3**-seq LRC is constructed by the following parity-check matrix H with girth **8**.

$$H = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

Erasures: $1^{st}, 2^{nd}, 7^{th}$ symbols

i^{th} symbol	Repair set
1	$\{\mathbf{2}, \mathbf{3}\}$ and $\{4,7\}$
2	$\{1,3\}$ and $\{\mathbf{5}, \mathbf{8}\}$
7	$\{1,4\}$ and $\{\mathbf{8}, \mathbf{9}\}$

Constructing a parity check matrix for QC-LDPC codes

exponent matrix $E = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 4 & 6 \end{pmatrix}$ over the integers and a positive integer m

parity check matrix $H = \begin{bmatrix} I^{(1)} & I^{(1)} & I^{(1)} & I^{(1)} \\ I^{(0)} & I^{(1)} & I^{(4)} & I^{(6)} \end{bmatrix}$ over GF(2) of size $2m \times 4m$

where $I^{(j)}$ is the $m \times m$ identity matrix whose columns are circularly shifted j times

Example: **Select $m = 13$** $\Rightarrow$ H becomes a 26×52 matrix with

$$I^{(0)} = I = \begin{array}{c} 1000000000000 \\ 0100000000000 \\ 0010000000000 \\ 0001000000000 \\ 0000100000000 \\ 0000010000000 \\ 0000001000000 \\ 0000000100000 \\ 0000000010000 \\ 0000000001000 \\ 0000000000100 \\ 0000000000010 \\ 0000000000001 \end{array} \quad I^{(1)} = \begin{array}{c} 0100000000000 \\ 0010000000000 \\ 0001000000000 \\ 0000100000000 \\ 0000010000000 \\ 0000001000000 \\ 0000000100000 \\ 0000000010000 \\ 0000000001000 \\ 0000000000100 \\ 0000000000010 \\ 0000000000001 \\ 1000000000000 \end{array} \quad I^{(4)} = \begin{array}{c} 0000100000000 \\ 0000010000000 \\ 0000001000000 \\ 0000000100000 \\ 0000000010000 \\ 0000000001000 \\ 0000000000100 \\ 0000000000010 \\ 0000000000001 \\ 1000000000000 \\ 0100000000000 \\ 0010000000000 \\ 0001000000000 \end{array} \quad I^{(6)} = \begin{array}{c} 0000001000000 \\ 0000000100000 \\ 0000000010000 \\ 0000000001000 \\ 0000000000100 \\ 0000000000010 \\ 0000000000001 \\ 1000000000000 \\ 0100000000000 \\ 0010000000000 \\ 0001000000000 \\ 0000100000000 \\ 0000010000000 \end{array}$$

[5] M.P.C. Fossorier, *IEEE Trans. Inf. Theory*, 2004 gives a necessary condition for the existence of even-cycles in H

$$E = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 4 & 6 \end{pmatrix} \quad \Rightarrow \quad H = \begin{bmatrix} I^{(1)} & I^{(1)} & I^{(1)} & I^{(1)} \\ I^{(0)} & I^{(1)} & I^{(4)} & I^{(6)} \end{bmatrix}$$

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0000000000001	1000000000000	0001000000000	0000010000000

We will see later that
this parity check matrix
gives

a 5-seq LRC
with parameters:
length $n = 52$,
dimension $k = 25$,
locality $r = 3$
and availability $t = 2$

What is this?

$$E = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 4 & 6 \end{pmatrix} \Rightarrow H = \begin{bmatrix} I^{(1)} & I^{(1)} & I^{(1)} & I^{(1)} \\ I^{(0)} & I^{(1)} & I^{(4)} & I^{(6)} \end{bmatrix}$$

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We will see later that **this parity check matrix** gives

a 5-seq LRC
with parameters:
length $n = 52,$
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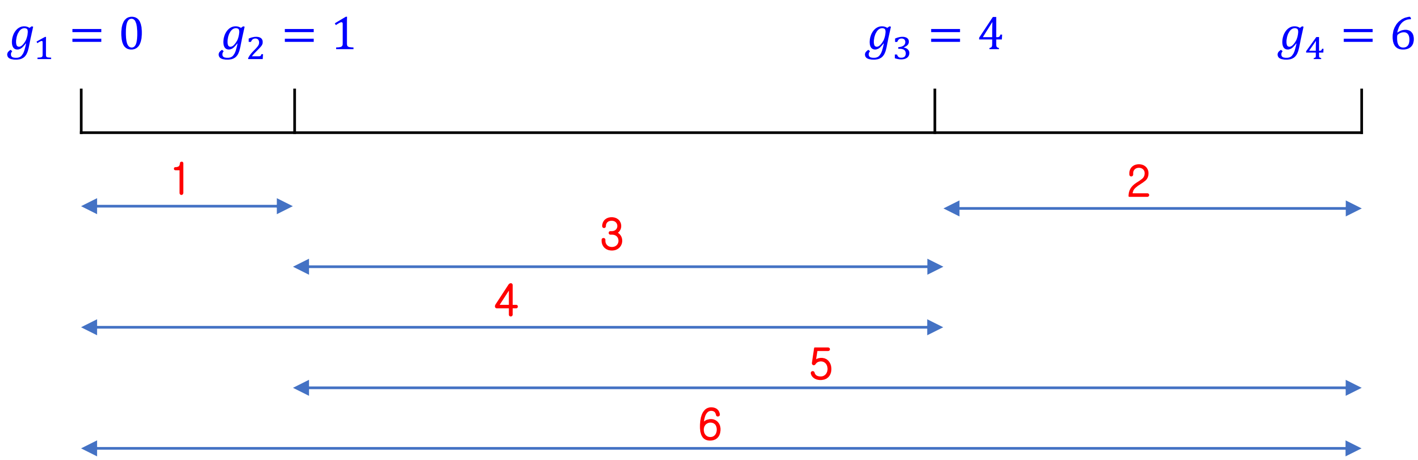
Golomb rulers

s-mark Golomb ruler:

A set of s integers $0 = g_1 < g_2 < \ldots < g_s$ such that $g_j - g_i$ for $i < j$ are all distinct.

Example: 4-mark Golomb ruler

$$\{ g_1, g_2, g_3, g_4 \} = \{ 0, 1, 4, 6 \}$$



Interval **distances** are all distinct

Contents

Main Result

Construction of 5-seq LRCs with availability $t = 2$ from a Golomb Ruler

- We need an s-mark Golomb ruler: $0 = g_1 < g_2 < \ \dots < g_s$ of length g_s.
  - distances are all distinct ⇒ $g_i \neq g_j$ for $i < j$

    as well as $g_j - g_i \neq g_l - g_k$ for $i < j$ and $k < l$ with $i \neq k$

- We need to choose a positive integer m satisfying the following three conditions:
  1) $g_i \neq g_j \pmod{m}$ for $i < j$
  2) $g_j - g_i \neq -(g_l - g_k) \pmod{m}$ for $i < j$ and $k < l$ but not necessarily $i \neq k$
  3) GCD of m and all the distances $(g_j - g_i)'s$ is collectively 1 (not pairwise)

⇐ $m > g_s$

⇐ $m > 2g_s$ and some more

⇐ any m when $g_j - g_i = 1$ for some $i < j$

Main Result

Construction of 5-seq LRCs with availability t=2 from a Golomb Ruler

Theorem 1.

- Let $0 = g_1 < g_2 < \dots < g_s$ of length g_s be an s-mark Golomb ruler.
- Let m be a positive integer satisfying the three conditions above.
- Let the exponent matrix E be of size $2 \times s$ in which the first row is a constant number c and the second row is $g_1, g_2, \dots, g_s$.
- Construct a binary $2m \times sm$ matrix H by substituting $I^{(c)}$ or $I^{(g_j)}$ into the positions of E.

Then, H is a parity check matrix of a **5-seq LRC** with
$n = sm, k = (s-2)m + 1,\ r = s - 1$ and availability $t = 2$.

Example

- 4-mark Golomb ruler: {0,1,4,6}
- Choose $m = 13 > 6 = g_s$ and $13 > 12 = 2g_s$
  - ✓ One of the distances is 1. So the third condition is automatically satisfied.

$$E = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 4 & 6 \end{pmatrix} \Rightarrow H = \begin{bmatrix} I^{(1)} & I^{(1)} & I^{(1)} & I^{(1)} \\ I^{(0)} & I^{(1)} & I^{(4)} & I^{(6)} \end{bmatrix}$$

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This parity check matrix
gives

a 5-seq LRC
with parameters:
length $n = 52$,
dimension $k = 25$,
locality $r = 3$
and availability $t = 2$

Some Remarks

This constant can be any number.

$$E = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 4 & 6 \end{pmatrix} \Rightarrow H = \begin{bmatrix} I^{(1)} & I^{(1)} & I^{(1)} & I^{(1)} \\ I^{(0)} & I^{(1)} & I^{(4)} & I^{(6)} \end{bmatrix}$$

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Some Remarks

This can be any Golomb ruler. The number of marks determines its size.

$$E = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 4 & 6 \end{pmatrix} \Rightarrow H = \begin{bmatrix} I^{(1)} & I^{(1)} & I^{(1)} & I^{(1)} \\ I^{(0)} & I^{(1)} & I^{(4)} & I^{(6)} \end{bmatrix}$$

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s-mark → length = sm

H matrix has dimension $2m \times sm$

$$n - k = 2m$$
or
$$k = n - 2m = sm - 2m = (s-2)m$$

when the rows of H are all linearly independent.

Otherwise, in general, we have

$$k \geq (s-2)m$$

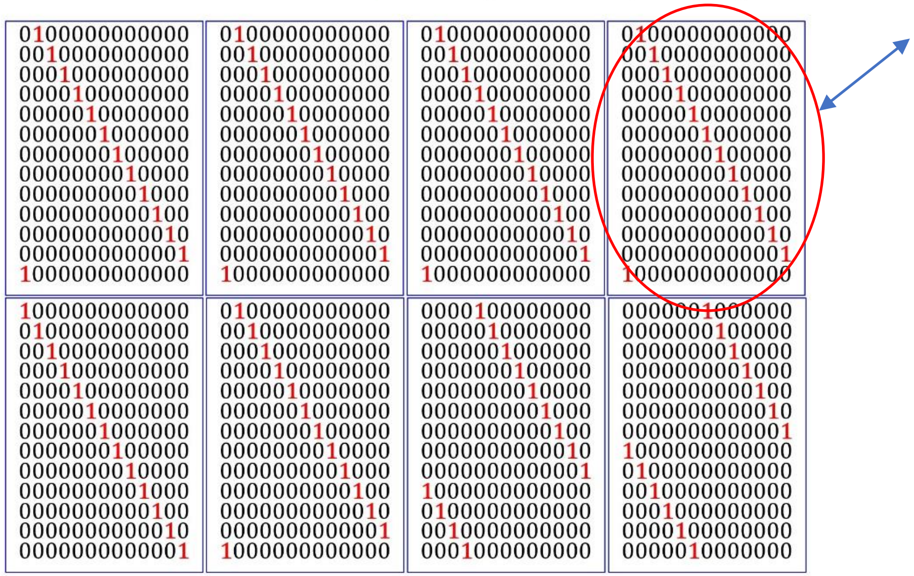
In this case, we have exactly

$$k = (s-2)m + 1$$

Some Remarks

- $g_i \neq g_j$ (mod m) for $i < j$
- $g_j - g_i \neq -(g_l - g_k)$ (mod m) for $i < j$ and $k < l$ but not necessarily $i \neq k$
- GCD of m and all the distances $(g_j - g_i)'s$ is collectively 1 (not pairwise)

$$E = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 4 & 6 \end{pmatrix} \Rightarrow H = \begin{bmatrix} I^{(1)} & I^{(1)} & I^{(1)} & I^{(1)} \\ I^{(0)} & I^{(1)} & I^{(4)} & I^{(6)} \end{bmatrix}$$



The size m must be carefully selected for a given s-mark Golomb ruler.

- Easy choice is $m > 2g_s$
- There are possible values of m in the range of $g_s < m < 2g_s$

- We found the proposed 5-seq LRC in some cases becomes **optimal** in the sense of the **code rate**.

Availability $t = 2$

$$E = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 4 & 6 \end{pmatrix} \Rightarrow H = \begin{bmatrix} I^{(1)} & I^{(1)} & I^{(1)} & I^{(1)} \\ I^{(0)} & I^{(1)} & I^{(4)} & I^{(6)} \end{bmatrix}$$

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This parity check matrix
gives

a 5-seq LRC
with parameters:
length $n = 52$,
dimension $k = 25$,
locality $r = 3$
and **availability $t = 2$**

We check ...

$$H = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}$$

with disjoint repair group

$$\left| supp(h_{1,i}) \cap supp(h_{2,j}) \right| \le 1, \text{ for all } i, j$$

where $h_{1,i}$ and $h_{2,j}$ are i-th row of H_1 and j-th row of H_2, respectively.

[1] H. Choi, Z. Jing, G. Kim and H. -Y. Song, "Some Intersections of two Binary LRCs with Disjoint Repair Groups," The 10th International Workshop on Signal Design and its Applications in Communications (IWSDA 2022), August, 2022.

Availability $t = 2$

$$E = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 4 & 6 \end{pmatrix} \Rightarrow H = \begin{bmatrix} I^{(1)} & I^{(1)} & I^{(1)} & I^{(1)} \\ I^{(0)} & I^{(1)} & I^{(4)} & I^{(6)} \end{bmatrix}$$

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This parity check matrix
gives

a 5-seq LRC
with parameters:
length $n = 52$,
dimension $k = 25$,
locality $r = 3$
and **availability $t = 2$**

First, consider any two distinct rows h_i, h_j **from the upper half**

$$|supp(h_i) \cap supp(h_j)| = 0$$

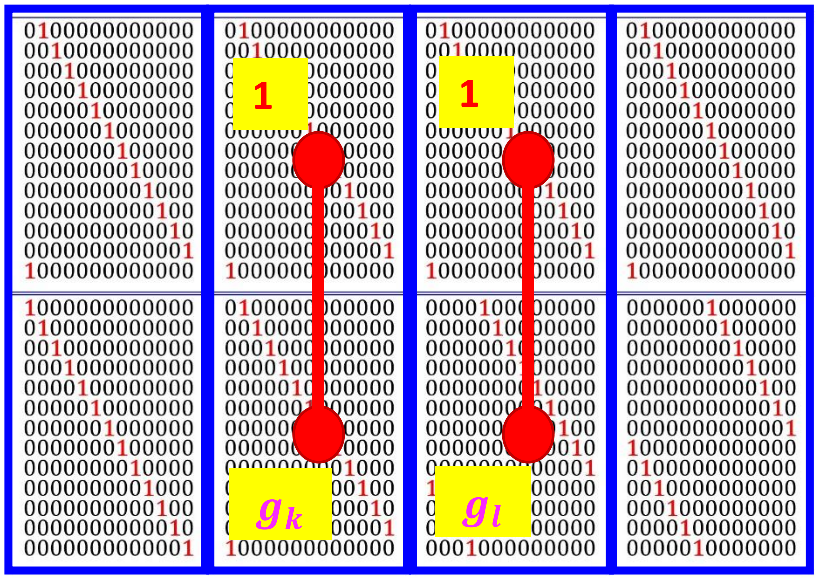
Similarly, the same is true for two distinct rows **from the lower half**.

Availability $t = 2$

$$E = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 4 & 6 \end{pmatrix} \Rightarrow H = \begin{bmatrix} I^{(1)} & I^{(1)} & I^{(1)} & I^{(1)} \\ I^{(0)} & I^{(1)} & I^{(4)} & I^{(6)} \end{bmatrix}$$

x-th row

y-th row



This parity check matrix gives

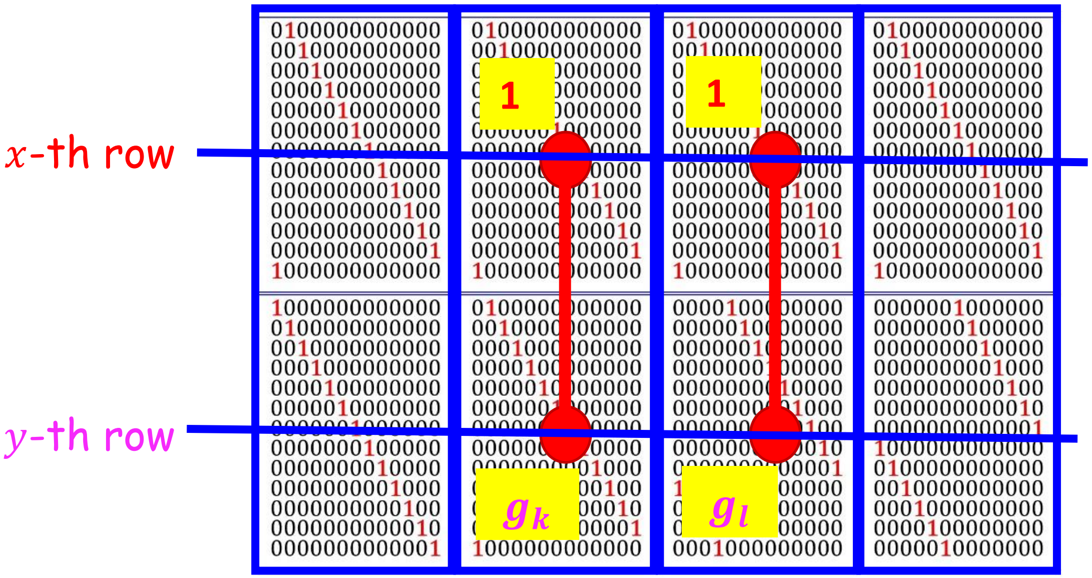
a 5-seq LRC
with parameters:
length $n = 52$,
dimension $k = 25$,
locality $r = 3$
and **availability $t = 2$**

Second, assume on the contrary that $|supp(h_{1,x}) \cap supp(h_{2,y})| \geq 2$

We find two 1s in the same coordinate, say in blocks $I^{(g_k)}$ and $I^{(g_l)}$ in the lower half, and the corresponding two blocks $I^{(1)}$ in the upper half.

Availability $t = 2$

$$E = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 4 & 6 \end{pmatrix} \Rightarrow H = \begin{bmatrix} I^{(1)} & I^{(1)} & I^{(1)} & I^{(1)} \\ I^{(0)} & I^{(1)} & I^{(4)} & I^{(6)} \end{bmatrix}$$



This parity check matrix
gives

a 5-seq LRC
with parameters:
length $n = 52$,
dimension $k = 25$,
locality $r = 3$
and **availability $t = 2$**

Second, assume on the contrary that $|supp(h_{1,x}) \cap supp(h_{2,y})| \geq 2$

We find two 1s in the same coordinate, say in blocks $I^{(g_k)}$ and $I^{(g_l)}$ in the lower half, and the corresponding two blocks $I^{(1)}$ in the upper half.

This implies that

$$1 - g_k \equiv x - y \equiv 1 - g_l \pmod m$$
or
$$g_k \equiv g_l \pmod m \text{ for } k \neq l.$$
(contradiction to the choice of m)

Examples of various 5-seq LRCs from Theorem 1

[2] S. B. Balaji, G. R. Kini, and P. V. Kumar, "A tight rate bound and a matching construction for locally recoverable codes with sequential recovery from any number of multiple erasures," in Proc. IEEE Int. Sym. Inf. Theory (ISIT), pp. 1778-1782, Jun. 2017.

- In [2], it was proposed that for $r \geq 3$, when u is odd and $\sigma = \left\lfloor \frac{u-1}{2} \right\rfloor$, the bound is as follows :

$$\frac{k}{n} \leq \frac{r^{\sigma+1}}{r^{\sigma+1}+2\sum_{i=1}^{\sigma} r^i+(u-2\sigma)} \quad (1)$$

?

TABLE I. EXAMPLES OF VARIOUS 5-SEQ LRCS FROM Theorem 1 USING OPTIMAL GOLOMB RULERS

s-mark Golomb rulers	smallest m that satisfies second condition in the range $m > g_s$	n	k	code rate	rate bound (1) for $u = 5$
0, 1, 4, 6	13	52	27	0.51923	0.51923
0, 1, 4, 9, 11	23	115	70	0.60870	0.60952
0, 2, 7, 8, 11	21	105	64	0.60952	
0, 1, 4, 10, 12, 17 0, 1, 8, 11, 13, 17	31	186	125	0.67204	0.67204
0, 2, 3, 10, 16, 21, 25 0, 2, 7, 13, 21, 22, 25	49	343	246	0.71720	0.71761
0, 1, 4, 9, 15, 22, 32, 34	69	552	415	0.75181	0.75219
0, 1, 5, 12, 25, 27, 35, 41, 44	89	801	624	0.77903	0.77930
0, 1, 6, 10, 23, 26, 34, 41, 53, 55	91	910	729	0.80110	0.80110

Contents

- Preliminary
- 5-seq LRCs with availability 2 from Golomb Rulers
- Optimal case
- Concluding Remark

Rate bound for sequential recovery LRCs

[2] S. B. Balaji, G. R. Kini, and P. V. Kumar, "A tight rate bound and a matching construction for locally recoverable codes with sequential recovery from any number of multiple erasures," in Proc. IEEE Int. Symp. Inf. Theory (ISIT), pp. 1778-1782, Jun. 2017.

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$$\frac{k}{n} \leq \frac{r^{\sigma+1}}{r^{\sigma+1} + 2\sum_{i=1}^{\sigma} r^i + (u - 2\sigma)} \qquad (1)$$

- For $r = s - 1$ and $u = 5$, the bound (1) becomes

$$\frac{k}{n} \leq \frac{r^3}{r^3 + 2(r + r^2) + 1} = \frac{s^3 - 3s^2 + 3s - 1}{s^3 - s^2 + s}$$

Rate bound for sequential recovery LRCs

[2] S. B. Balaji, G. R. Kini, and P. V. Kumar, "A tight rate bound and a matching construction for locally recoverable codes with sequential recovery from any number of multiple erasures," in Proc. IEEE Int. Sym. Inf. Theory (ISIT), pp. 1778-1782, Jun. 2017.

- For $r = s-1$ and $u = 5$, the bound (1) becomes

$$\frac{k}{n} \leq \frac{r^3}{r^3+2(r+r^2)+1} = \frac{(s-1)^3}{s^3-s^2+s} = \frac{(s-1)^3}{s(s^2-s+1)}$$

- The code rate derived from **Theorem 1** becomes

$$\frac{k}{n} = \frac{(s-2)m+1}{sm}$$

- Therefore, for the optimal case, we have

$$\frac{(s-1)^3}{s(s^2-s+1)} = \frac{(s-2)m+1}{sm} \quad \text{or} \quad m = s^2 - s + 1$$

- Therefore, the question now becomes:

  Does there exist an s-mark Golomb ruler such that the positive integer $m = s^2 - s + 1$ satisfies the **three conditions** in Theorem 1?

  - $g_i \neq g_j$ (mod m) for $i < j$
  - $g_j - g_i \neq -(g_l - g_k)$ (mod m) for $i < j$ and $k < l$ but not necessarily $i \neq k$
  - GCD of m and all the distances $(g_j - g_i)$'s is collectively 1 (not pairwise)

  $\Leftarrow m > g_s$

  $\Leftarrow m > 2g_s$ and some more

  $\Leftarrow$ any m when $g_j - g_i = 1$ for some $i < j$

Singer Difference sets

Does there exist an s-mark Golomb ruler such that the positive integer $m = s^2 - s + 1$ satisfies the **three conditions** in **Theorem 1**?

[3] J. Singer, "A theorem in finite projective geometry and some applications to number theory," Trans. Amer. Math. Soc., vol. 43, no. 3, pp. 377–385, 1938.

- One of the well-known planar difference set comes from Singer with parameters $(v = q^2 + q + 1,\ k = q + 1,\ \lambda = 1)$ when q is a power of prime [2].

m

number of integers in set

$= s$

$$m = s^2 - s + 1$$

$$= (q+1)^2 - (q+1) + 1 = q^2 + q + 1 = v$$

Singer Difference sets

[3] J. Singer, "A theorem in finite projective geometry and some applications to number theory," Trans. Amer. Math. Soc., vol. 43, no. 3, pp. 377–385, 1938.
[4] C. J. Colbourne and J. H. Dinitz, Handbook of Combinatorial Designs, 2nd ed. Boca Raton, FL, USA: CRC Press, 2007.

- One of the well-known planar difference set comes from Singer with parameters $(v = q^2 + q + 1,\ k = q + 1,\ \lambda = 1)$ when q is a power of prime [2].

Construction (18.28 in [3]) $\boldsymbol{Singer\ difference\ sets}$.

Let α be a generator of the multiplicative group of F_{q^n}.
Then the set of integers $\left\{i : 0 \le i < \frac{q^n-1}{q-1},\ \ \mathrm{trace}_{n/1}(\alpha^i) = 0\right\}$ modulo $\frac{q^n-1}{q-1}$
form a (cyclic) difference set.
Here, the *trace* denotes the usual trace function

$$\mathrm{trace}_{n/1}(\beta) = \sum_{i=0}^{n-1} \beta^{q^i}$$

from F_{q^n} onto F_q.

Example: (13, 4, 1) Singer difference set

Golomb rulers from Singer Difference sets

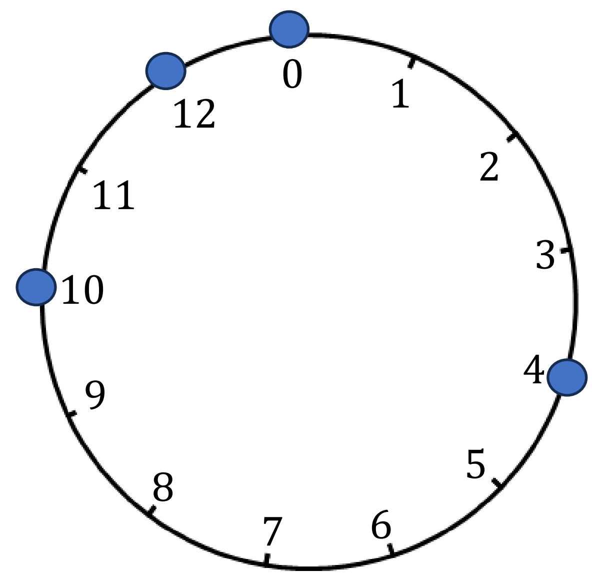
Example: (13, 4, 1) Singer difference set

Find a set of integers $\{i : 0 \leq i < 13, \ \ \mathrm{trace}_{3/1}(\alpha^i) = 0\}$ modulo 13

$GF(3^3) = F_{27} = F_3(\alpha)$ using α which is a root of $x^3 + 2x^2 + 1$

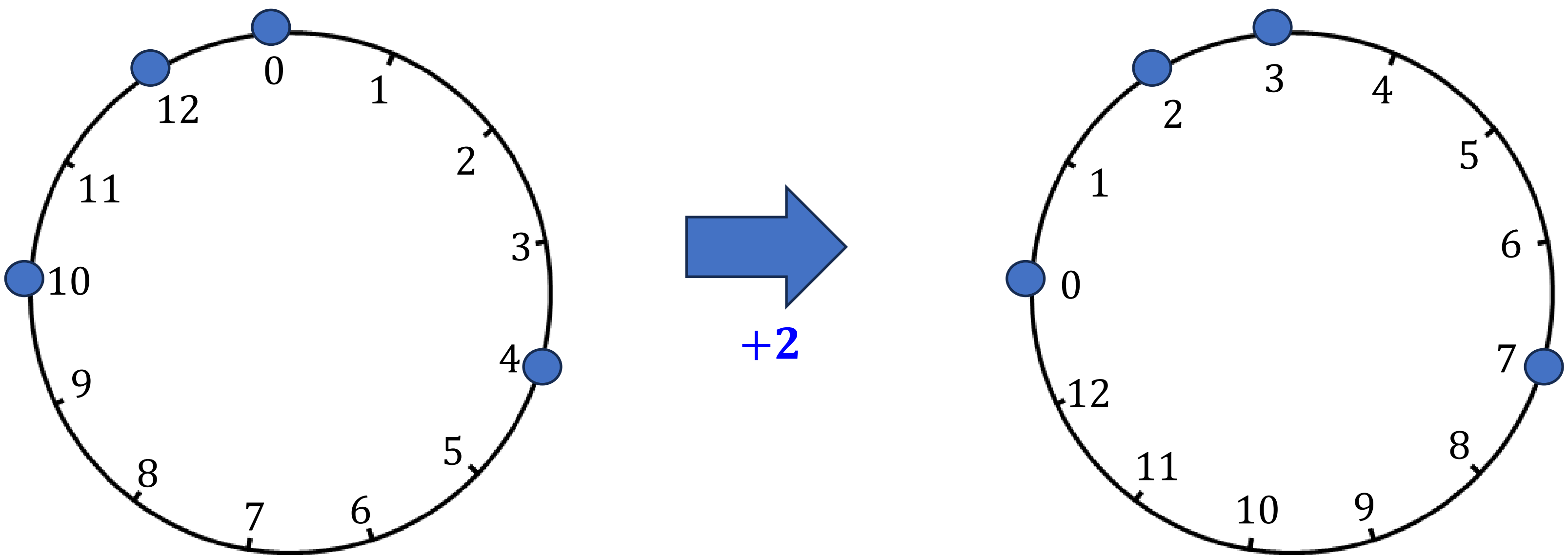
i	α^i	$\mathrm{Tr}(\alpha^i)$
*	0	0
0	1	0
1	α	1
2	α^2	1
3	$\alpha^2 + 2$	1
4	$\alpha^2 + 2\alpha + 2$	0
5	$2\alpha + 2$	2
6	$2\alpha^2 + 2\alpha$	1
7	$\alpha^2 + 1$	1
8	$\alpha^2 + \alpha + 2$	2
9	$2\alpha^2 + 2\alpha + 2$	1
10	$\alpha^2 + 2\alpha + 1$	0
11	$\alpha + 2$	1
12	$2\alpha^2 + 2\alpha$	0
	⋮	
25	$2\alpha^2 + 2$	0

$\{\mathbf{0}, \mathbf{4}, \mathbf{10}, \mathbf{12}\}$ **is a (13, 4, 1) Singer difference set**



Golomb rulers from Singer Difference sets

a-Shifts and t-Multiples are also difference sets
for any a mod v and any t mod v with (t,v)=1



$\{\mathbf{0}, \mathbf{4}, \mathbf{10}, \mathbf{12}\}$

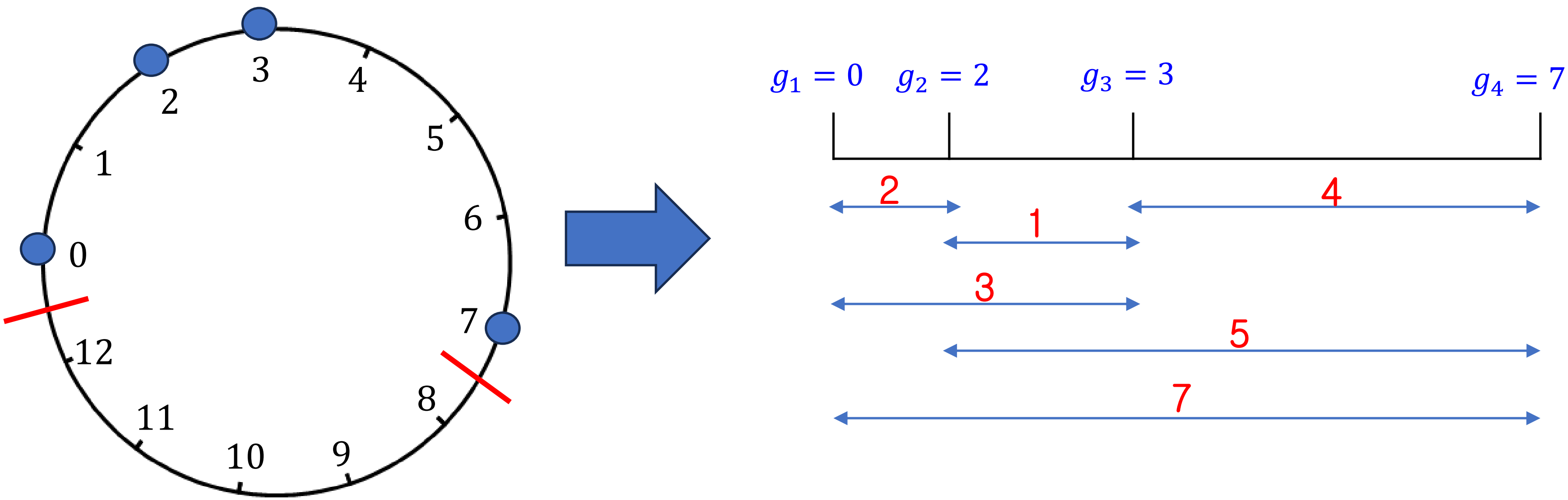
$\{\mathbf{0}, \mathbf{2}, \mathbf{3}, \mathbf{7}\}$

Golomb rulers from Singer Difference sets

(**example**) (13, 4, **1**) Singer difference set

$\{\mathbf{0}, \mathbf{2}, \mathbf{3}, \mathbf{7}\}$ **is a (13, 4, 1) Singer difference set**

4-mark Golomb ruler {0,2,3,7}



integers mod 13

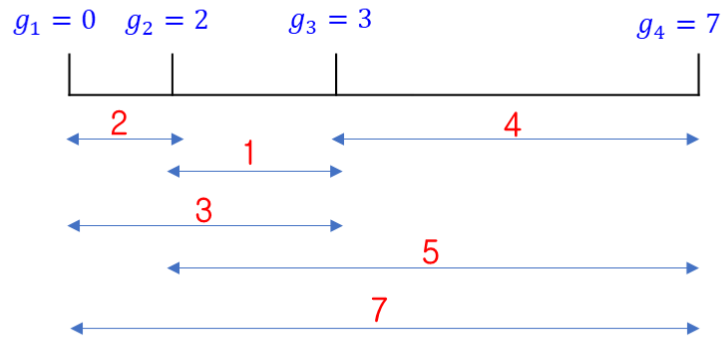
straight integers

Golomb rulers from Singer Difference sets

(**example**) (13, 4, **1**) Singer difference set

–	**0**	**2**	**3**	**7**
0	0	11	10	6
2	2	0	12	8
3	3	1	0	9
7	7	5	4	0

4-mark Golomb ruler {0,2,3,7}



The **sum of any two** of them can never be **13**.

second condition is satisfied !

- $g_j - g_i \neq -(g_l - g_k)$ (mod m) for $i < j$ and $k < l$ but not necessarily $i \neq k$

Examples - Optimality

TABLE II. EXAMPLES OF VARIOUS OPTIMAL 5-SEQ LRCS FROM **Theorem 1** USING GOLOMB RULERS FROM SINGER DIFFERENCE SETS

s	Golomb rulers from Singer difference sets [3][4]	m	n	k	code rate = rate bound (1)
4	0, 2, 3, 7	$g_s < 13 < 2g_s$	52	27	0.51923
	0, 1, 4, 6	$2g_s < 13$			
5	0, 2, 7, 8, 11	$g_s < 21 < 2g_s$	105	64	0.60952
6	0, 1, 4, 10, 12, 17	$g_s < 31 < 2g_s$	186	124	0.67204
	0, 1, 3, 8, 12, 18				
8	0, 1, 3, 13, 32, 36, 43, 52	$g_s < 57 < 2g_s$	456	343	0.75219
	0, 4, 5, 17, 19, 25, 28, 35				
9	0, 1, 3, 7, 15, 31, 36, 54, 63	$g_s < 73 < 2g_s$	657	512	0.77930
	0, 2, 10, 24, 25, 29, 36, 42, 45				

$$n = sm$$
$$k = (s-2)m + 1$$
$$r = s - 1$$
$$t = 2.$$

Contents

Concluding Remark

In this paper,

- We propose some new **5-seq LRCs with availability t = 2** based on **Golomb rulers.**
- The proposed **5-seq LRCs** can repair up to 5 erased symbols within **3 repair time**.
- We proved that the above codes are **rate-optimal** when the Golomb rulers derived from **Singer difference sets** are appropriately used.

We are about to submit the manuscript of this presentation and many more into some journal.
Therefore, please do not record this presentation, please!

Thank you for listening