The Unique Form of the Uncorrelated Optimal ZCZ Sequence Families

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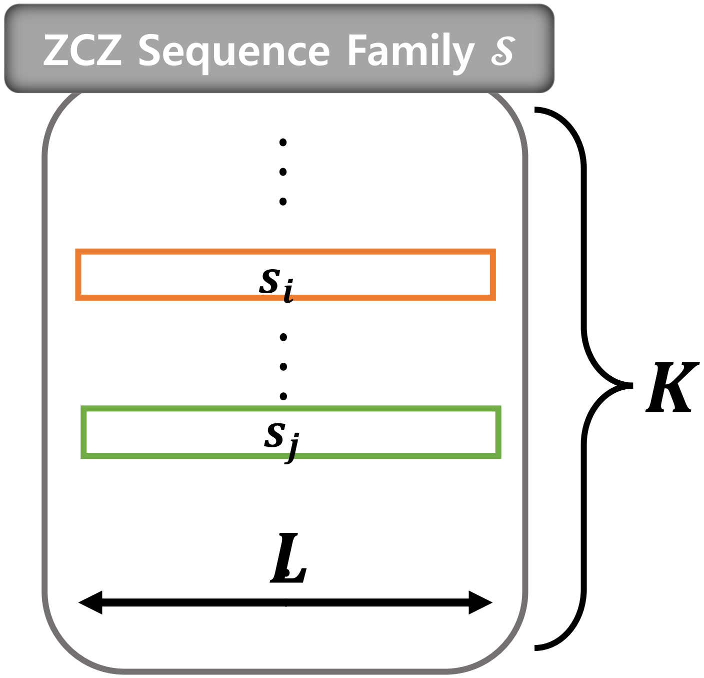
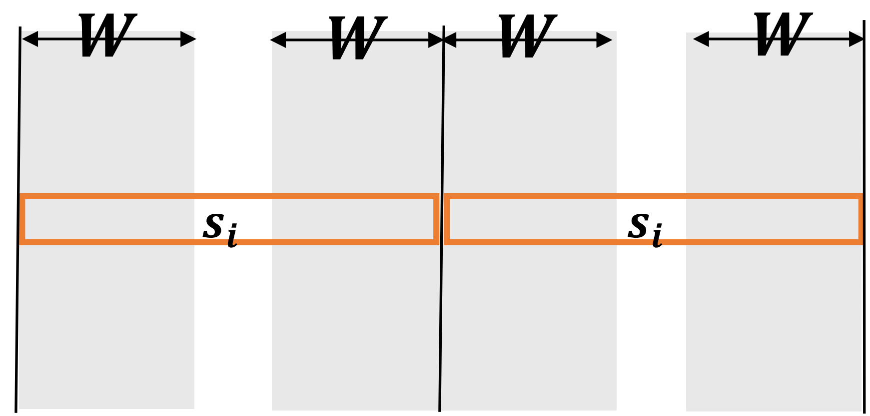
ZCZ Sequence Family



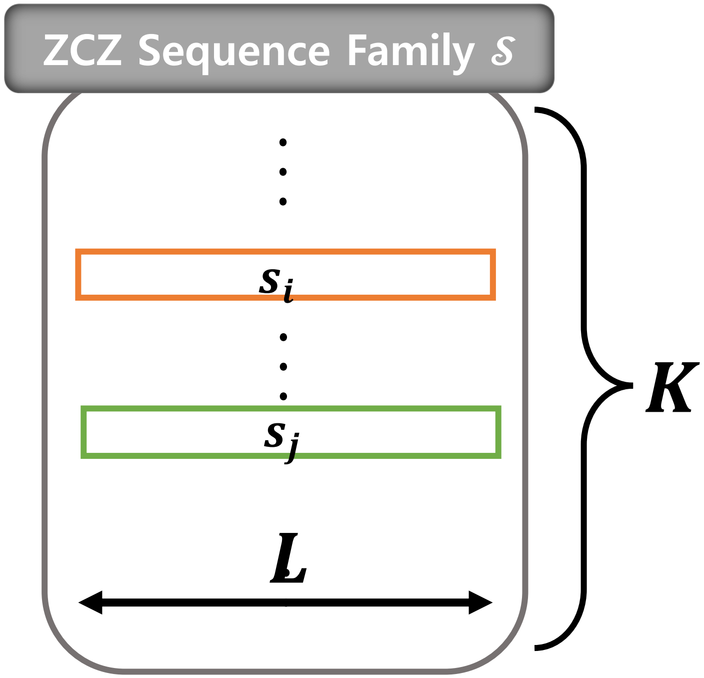


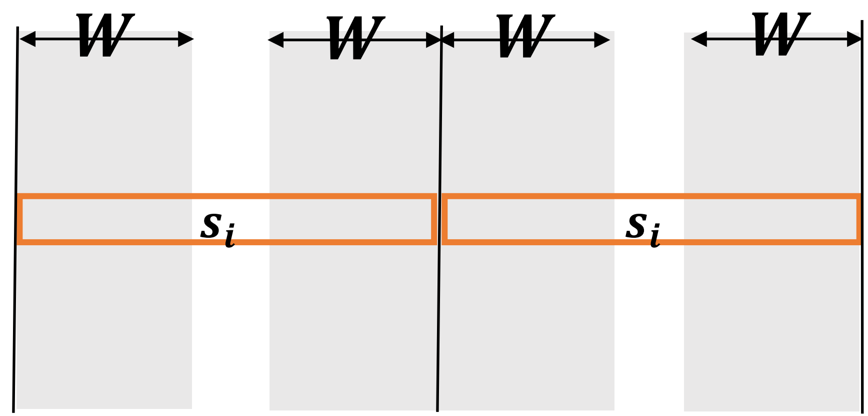
Well-known Theorem (Tang et. al. Elec.Lett.00**)**

For an (L, K, W)-ZCZ sequence family, we have

$$K \leq L/W$$

ZCZ Sequence Family





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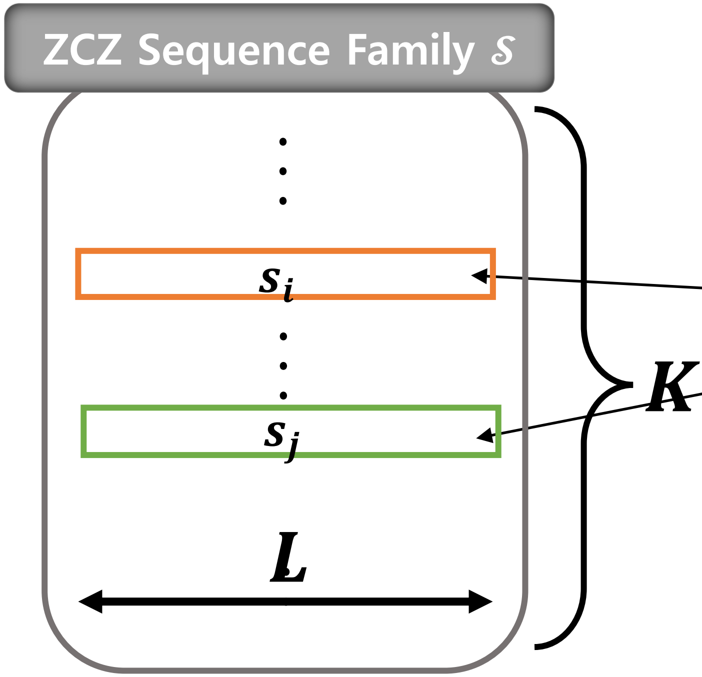
optimal ZCZ sequence family

Uncorrelated ZCZ Sequence Family



Uncorrelated:

The crosscorrelation is zero for any time shift

Well-known Theorem (Tang et. al. Elec.Lett.00**)**

For an (L, K, W)-ZCZ sequence family, we have

$$K \leq L/W$$

optimal ZCZ sequence family

Known (previous) constructions of the **uncorrelated optimal ZCZ Sequences**

Brodzik, polyphase (by ZaK transform) - IEEE TIT 2013

Brodzik, polyphase (by DFT) - Exc.Harm.Ana. 2015

Popović, **multi-level polyphase (from perfect sequence) - IEEE TIT 2018**

* he proved that Brodzik's constructions are special cases of this

Zhang, polyphase (from Mow's unified construction) - IEEE ISIT 2019

Fang and Wang, polyphase - IEEE ICICN 2022

* with **small** alphabet size, based on the Popović's construction

Kim (myself) et al, polyphase - IEEE ISIT 2023

* with **minimum** alphabet size, assuming the truth of Mow's conjecture

* and using the Popović's construction

Contribution of this Talk

Main Theorem:

any uncorrelated optimal ZCZ sequence family
can be constructed
from the Popovic's Construction:

Review of the Popović's Construction

Let K and W be positive integers

and ω_N be a complex primitive N th root of unity.

Let $\boldsymbol{g}_0, \boldsymbol{g}_1, \dots, \boldsymbol{g}_{K-1}$ be (not necessarily distinct) perfect sequences

all of length W.

Construct sequences $\boldsymbol{b}_0, \boldsymbol{b}_1, \dots, \boldsymbol{b}_{K-1}$ all of length KW as follows:

$$b_i(t) \triangleq g_i\big(t\ (mod\ W)\big)\omega_{KW}^{it}$$

for $t = 0,1,\dots,KW-1$

Then, $\mathcal{B} = \{\boldsymbol{b}_0, \boldsymbol{b}_1, \dots, \boldsymbol{b}_{K-1}\}$ is a uncorrelated optimal ZCZ sequence family with ZCZ parameter $(L = KW, K, W)$

Sketch of Proof

- STEP 1: Auto and Cross Correlation of **optimal** ZCZ sequences

  - correlated or uncorrelated

- STEP 2: Autocorrelation of **uncorrelated** optimal ZCZ sequences

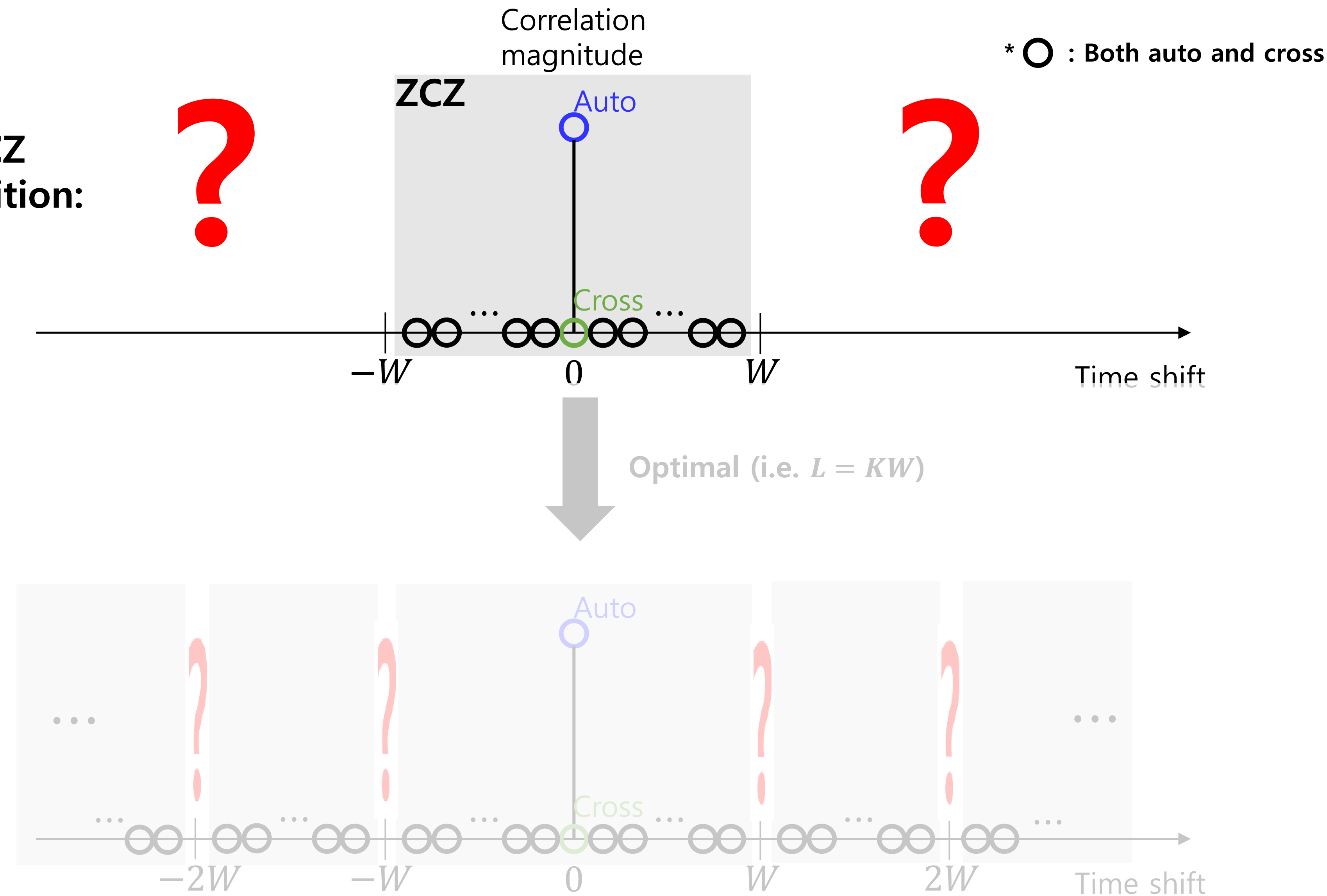
- STEP 3: Shape of **uncorrelated** optimal ZCZ sequences

  using Frequency domain approach

- STEP 4: Final Step

Auto and Cross of **optimal** ZCZ sequences
- correlated or uncorrelated

ZCZ Definition:



Auto and Cross of **optimal** ZCZ sequences
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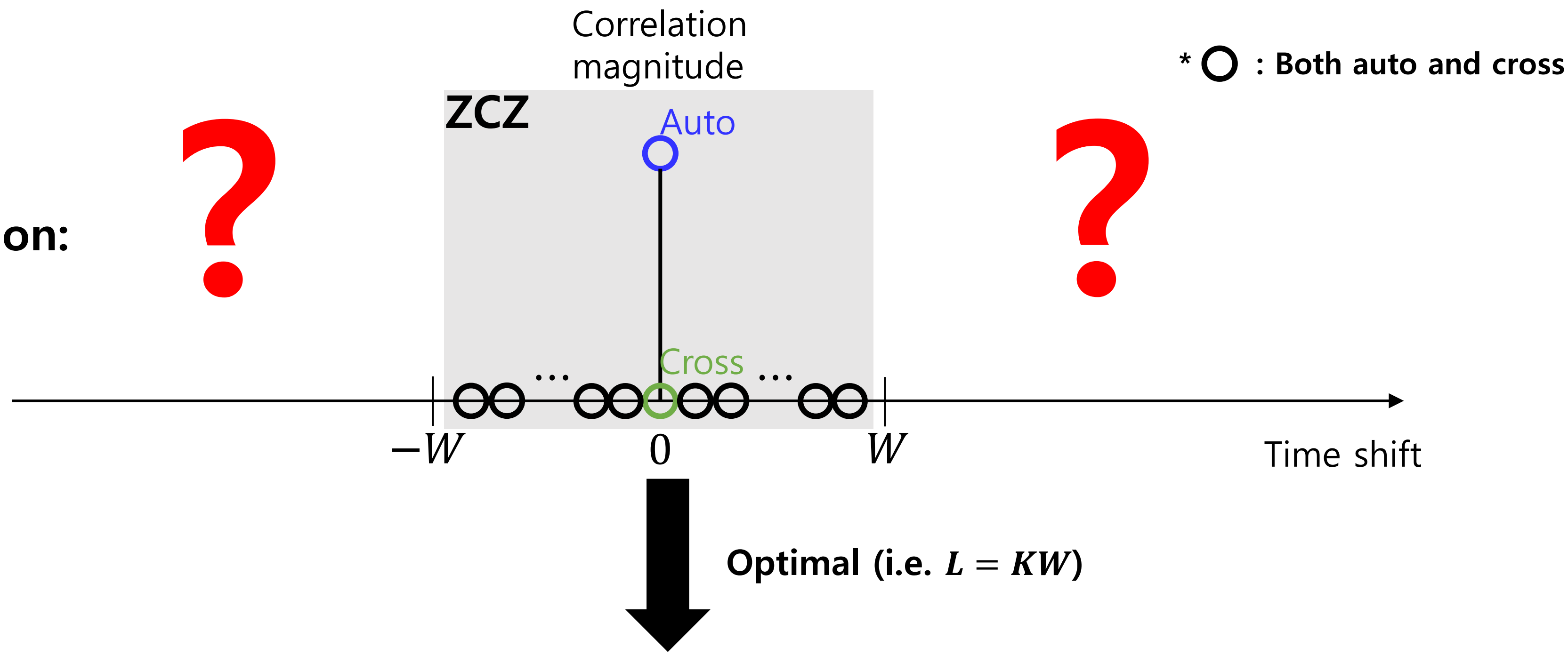

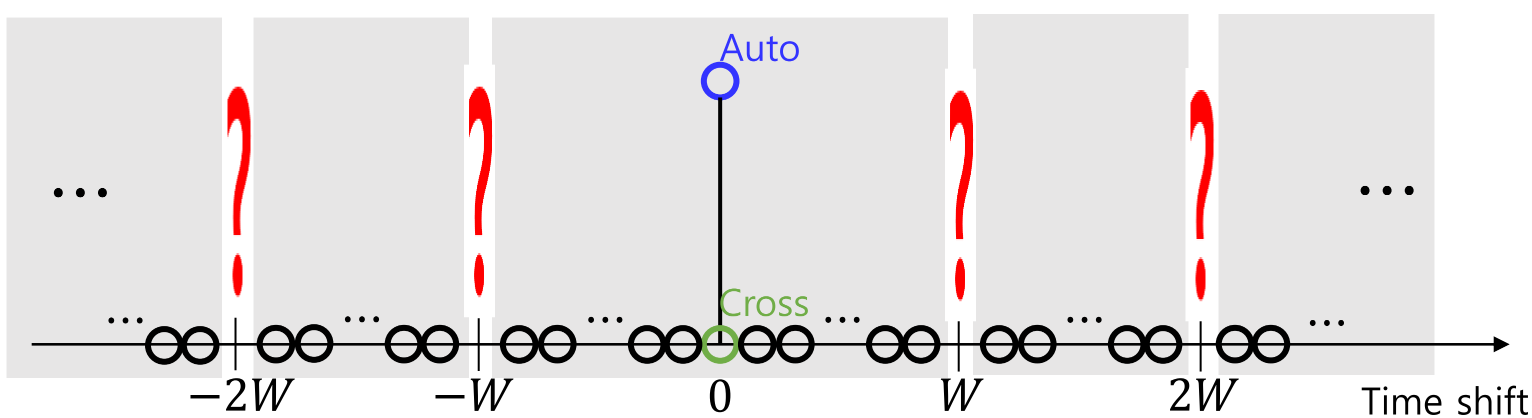

Auto and Cross of **optimal** ZCZ sequences
- correlated or uncorrelated

Consider an optimal $(L = KW, K, W)$-ZCZ sequence family

$$\mathcal{R} = \{\boldsymbol{r}_0, \boldsymbol{r}_1, \dots, \boldsymbol{r}_{K-1}\}.$$

Then, for any i *and* j (including the case of $i = j$)

$$C_{i,j}(\tau) = 0 \quad \text{if} \quad \tau \notin W\mathbb{Z}_{KW}$$

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- This result works when the optimal ZCZ family is **correlated.**
- When **uncorrelated**, the crosscorrelation would be trivial.
- The autocorrelation is **not trivial** even for **uncorrelated** optimal ZCZ.

Sketch of Proof

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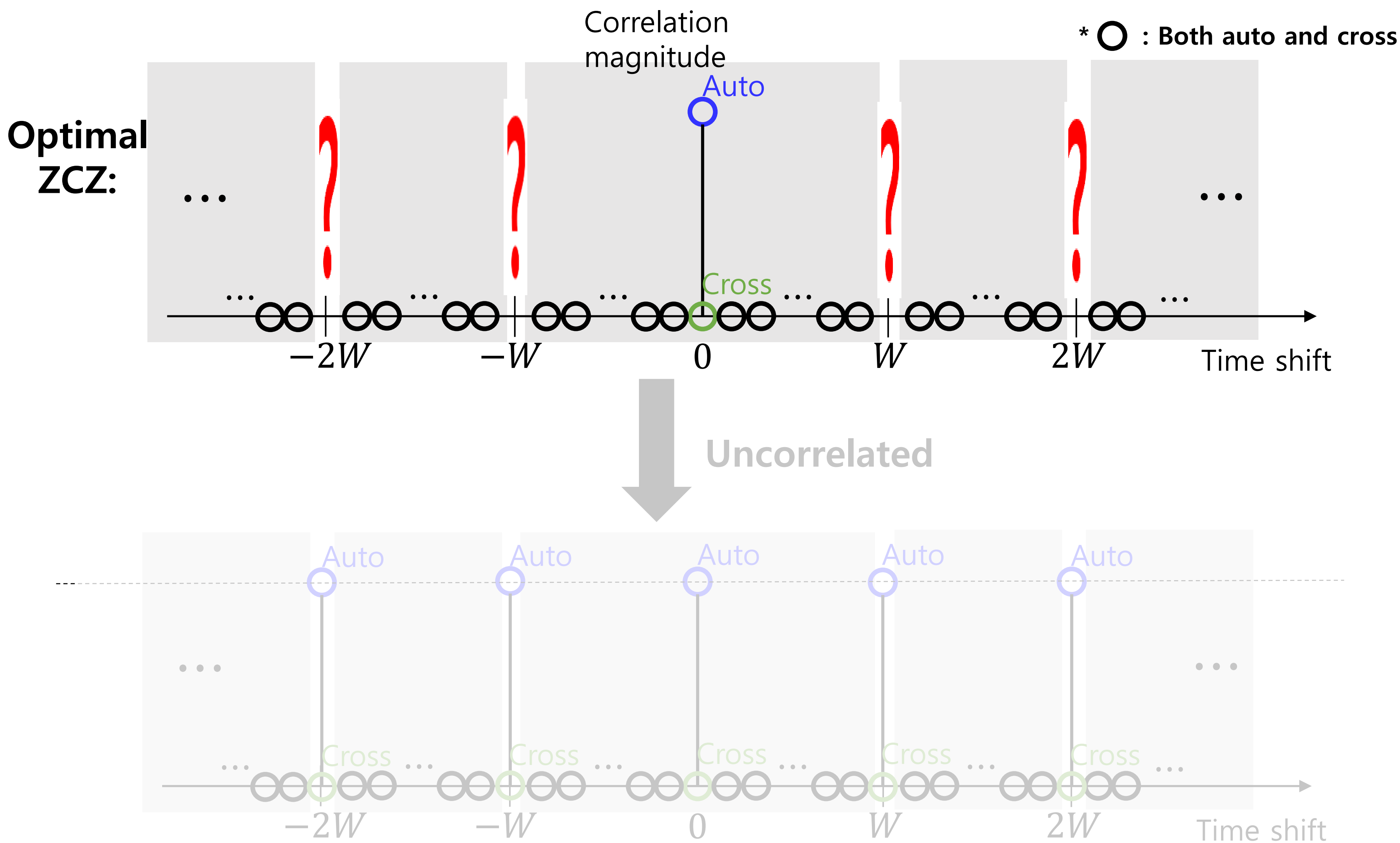
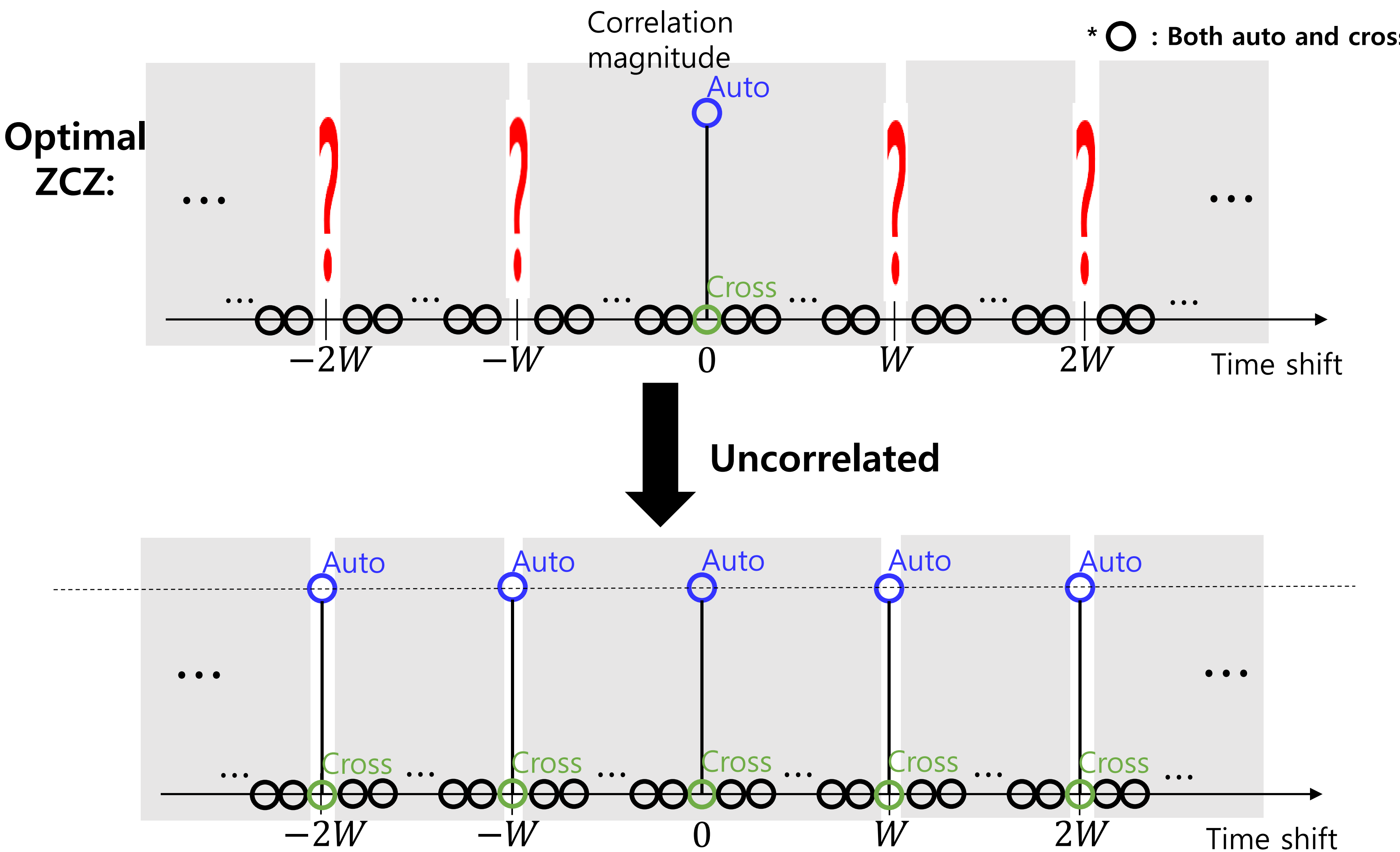
Autocorrelation of **uncorrelated** optimal ZCZ sequences



Autocorrelation of **uncorrelated** optimal ZCZ sequences



Autocorrelation of **uncorrelated** optimal ZCZ sequences

Consider a uncorrelated $(L = KW, K, W)$-ZCZ sequence family

$$\mathcal{R} = \{\boldsymbol{r}_0, \boldsymbol{r}_1, \dots, \boldsymbol{r}_{K-1}\}$$

For $i = 0, 1, \dots, K-1$, there exist an integer $\kappa(i)$ such that:

$$\boldsymbol{r}_{\boldsymbol{i}}(\boldsymbol{t} + \boldsymbol{W}) = \boldsymbol{\omega}_{\boldsymbol{K}}^{\boldsymbol{\kappa}(\boldsymbol{i})} \boldsymbol{r}_{\boldsymbol{i}}(\boldsymbol{t}) \textbf{ for } \boldsymbol{t} \in \mathbb{Z}_{\boldsymbol{KW}}$$

Autocorrelation of **uncorrelated** optimal ZCZ sequences

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$$\boldsymbol{r_i(t+W) = \omega_K^{\kappa(i)} r_i(t)} \textbf{ for } \boldsymbol{t} \in \mathbb{Z}_{\boldsymbol{KW}}$$

i.e.

$$\boldsymbol{C_i(\tau)} = \begin{cases} \boldsymbol{C_i(0)\omega_{KW}^{\kappa(i)\tau}}, & \textbf{for } \boldsymbol{\tau} \in \boldsymbol{W}\mathbb{Z}_{\boldsymbol{KW}} \\ \boldsymbol{0}, & \textbf{otherwise} \end{cases}$$

Sketch of Proof

- STEP 1: Auto and Cross Correlation of **optimal** ZCZ sequences

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Shape of sequences using Frequency domain approach

Correlation Theorem

$$\boldsymbol{C_i}(\cdot) \xrightarrow{\textbf{DFT}} \sqrt{KW} \times |\mathbf{DFT}_i(\cdot)|^2$$

Recall

$$\boldsymbol{C_i(\tau)} = \begin{cases} \boldsymbol{C_i(0)\,\omega_{KW}^{\kappa(i)\tau}}, & \textbf{for } \boldsymbol{\tau} \in \boldsymbol{W}\mathbb{Z}_{\boldsymbol{KW}} \\ \mathbf{0}, & \textbf{otherwise} \end{cases}$$

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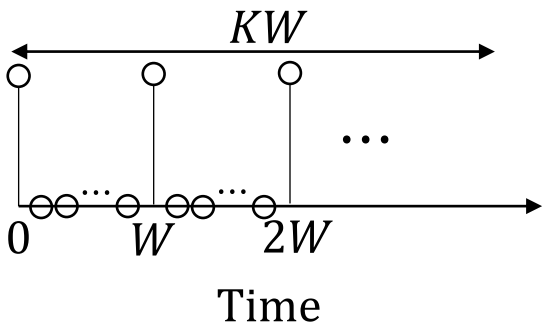

$\times$

$\boldsymbol{\omega_{KW}^{\kappa(i)\tau}}$

$$\xrightarrow{\textbf{DFT}} \quad ?$$

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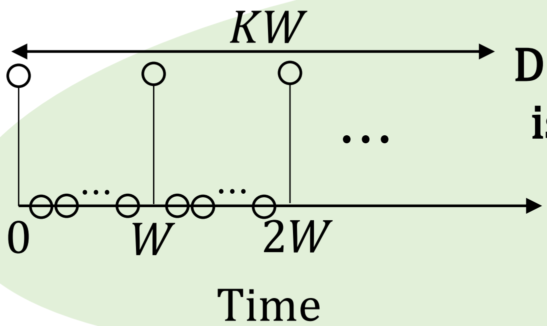

DFT of the impulse train is also an impulse train

$$\times$$

$$\boldsymbol{\omega_{KW}^{\kappa(i)\tau}}$$

This is the $-\boldsymbol{\kappa}(\boldsymbol{i})$ frequency shift operator

$$\xrightarrow{\textbf{DFT}} \quad ?$$

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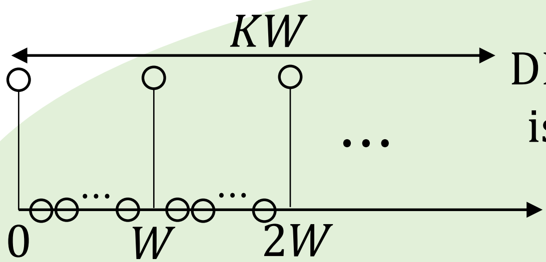
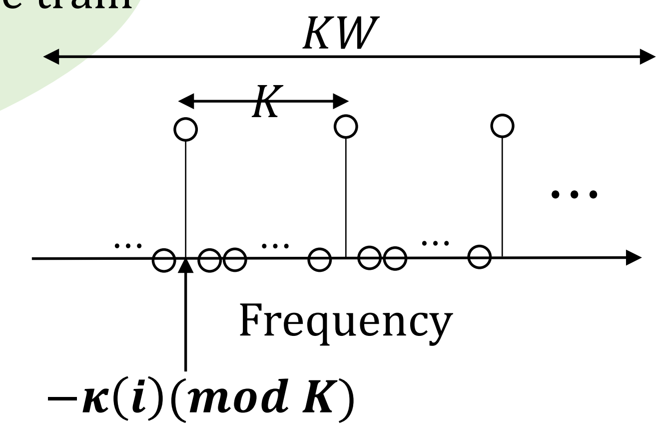


DFT of the impulse train is also an impulse train

DFT



×

$$\boldsymbol{\omega_{KW}^{\kappa(i)\tau}}$$

This is the $-\kappa(i)$ frequency shift operator

$$C_i(\tau) = \begin{cases} C_i(0)\omega_{KW}^{\kappa(i)\tau}, & \text{for } \tau \in W\mathbb{Z}_{KW} \\ 0, & \text{otherwise} \end{cases}$$

Shape of sequences using Frequency domain approach

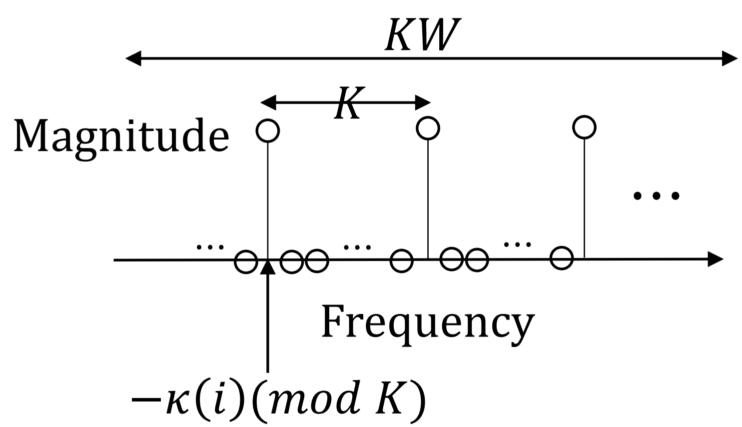
Time Domain		**Frequency Domain**
Uncorrelated	$\Longleftrightarrow$	Indexes of non-zero values are disjoint from each other

Shape of sequences using Frequency domain approach

Time Domain ⟷ **Frequency Domain**

Uncorrelated ⟷ Indexes of non-zero values are disjoint from each other

Recall the DFT shape of r_i



Shape of sequences using Frequency domain approach

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Uncorrelated

Frequency Domain

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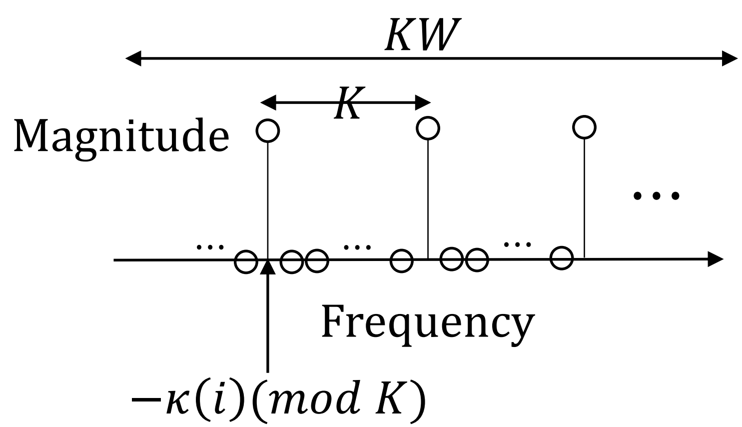

Therefore, $\kappa(i)\ (mod\ K)$ for $i = 0, 1, \dots, K-1$ are distinct with each other. That is,

$$\{\kappa(i)\ (mod\ K) \mid i = 0, 1, \dots, K-1\} = \{0, 1, \dots, K-1\}$$

Shape of sequences using Frequency domain approach

$\therefore$ WLOG, for all $\boldsymbol{i}$,

$$\boldsymbol{r_i}(t+\boldsymbol{W}) = \boldsymbol{\omega_K^i}\boldsymbol{r_i}(t) \ \text{ for } t = 0,1,\ldots,KW-1.$$

Then,

$$\boldsymbol{r}_i(t) = \boldsymbol{g}_i\big(t\ (mod\ \boldsymbol{W})\big)\left(\boldsymbol{\omega}_{\boldsymbol{KW}}^{+\boldsymbol{i}}\right)^t \ \text{ for } t = 0, 1,\ldots, KW-1,$$

where we may define

$$\boldsymbol{g}_i(t) \triangleq \left(\boldsymbol{\omega}_{\boldsymbol{KW}}^{-\boldsymbol{i}}\right)^t \boldsymbol{r}_i(t) \quad \text{for } t = 0,1,\ldots,W-1.$$

Now, we claim that

g_i **must be a perfect sequence**

which is the end of the proof.

Shape of sequences using Frequency domain approach

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Final Claim that g_i must be a perfect sequence

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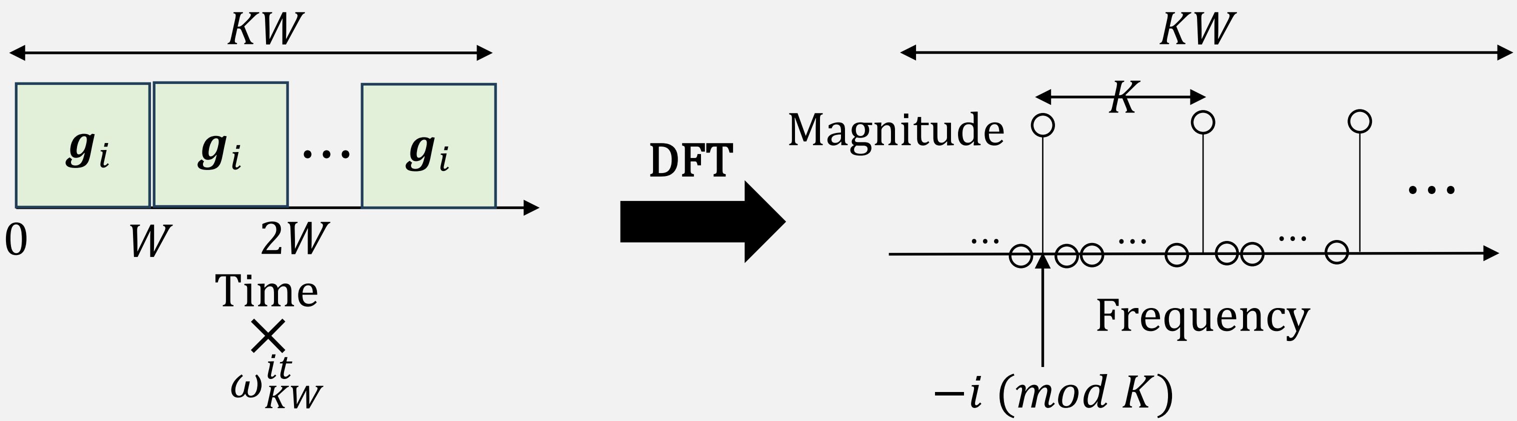
Now, we claim that

above g_i must be a perfect sequence

Proof: g_i must be a perfect sequence (1/5)

Recall that...

$r_i(t) = g_i\big(t \ (mod\ W)\big)\omega_{KW}^{it}$



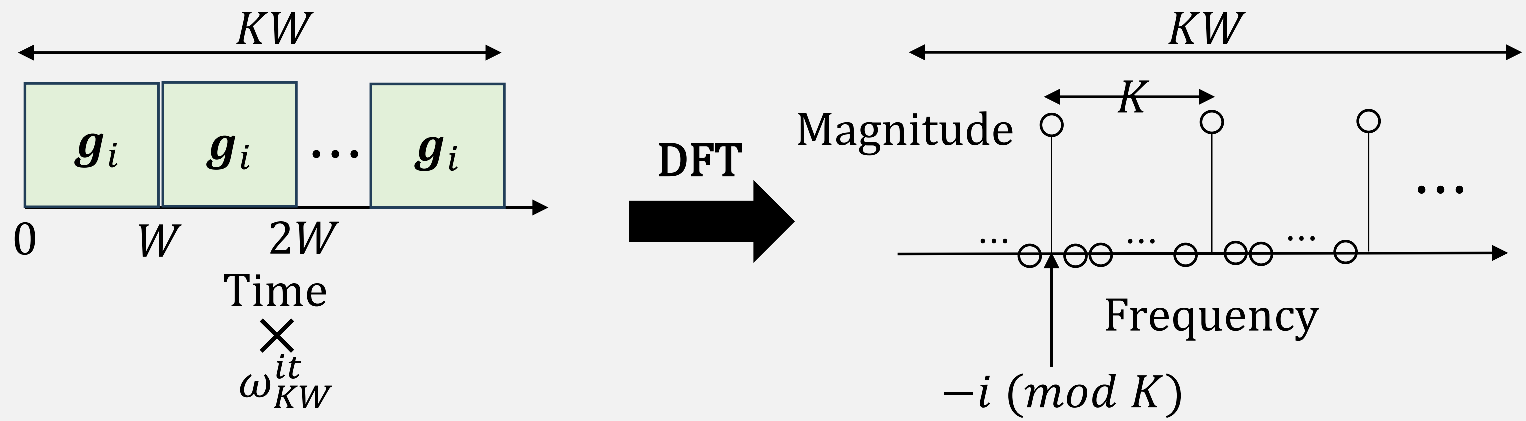
Now, we know roughly what r_i looks like in both time domain and frequency domain.

From both of those shapes, let's prove that g_i is a perfect sequence.

Proof : g_i must be a perfect sequence (2/5)

$r_i(t) = g_i\big(t \ (mod\ W)\big)\omega_{KW}^{it}$

Recall that...



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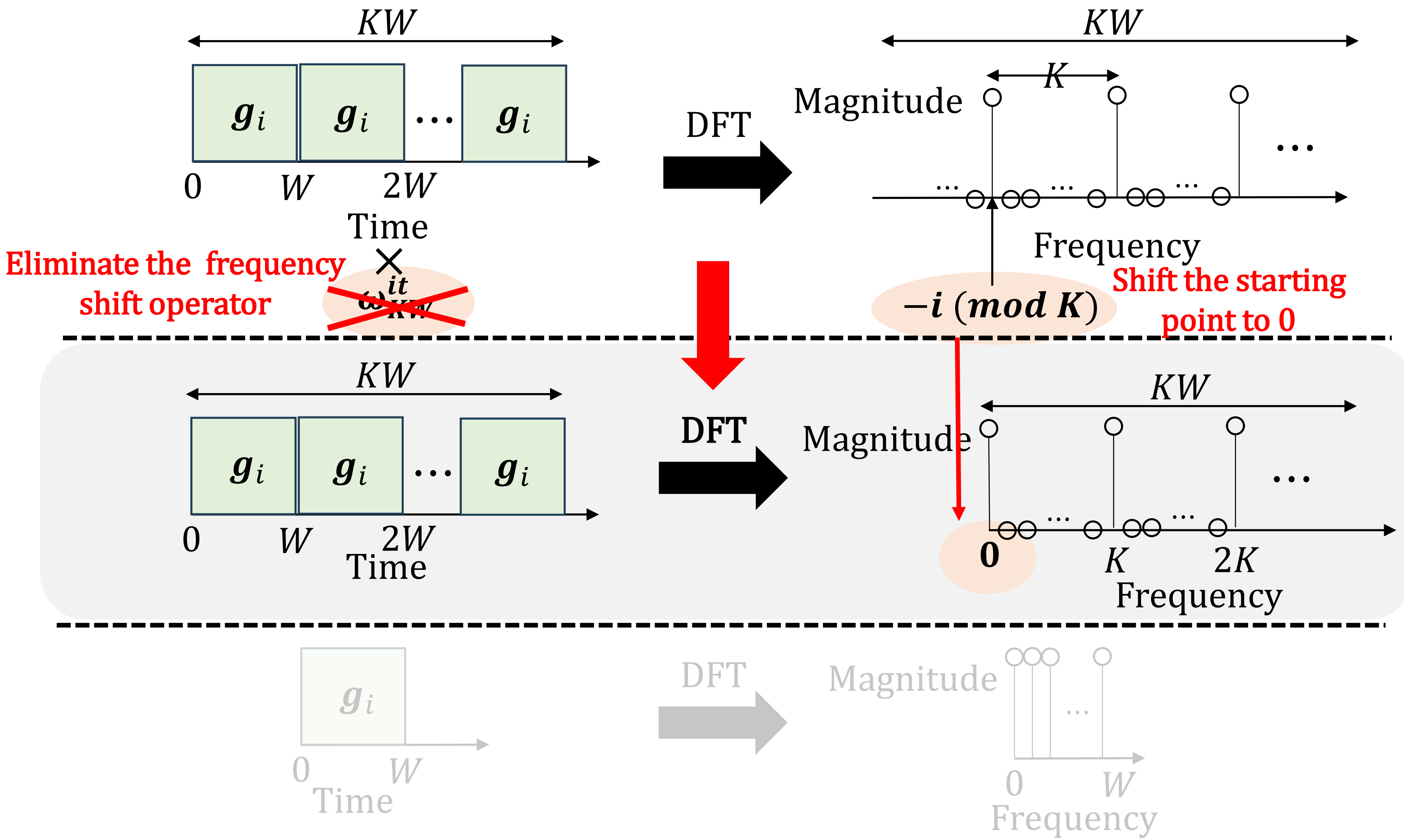
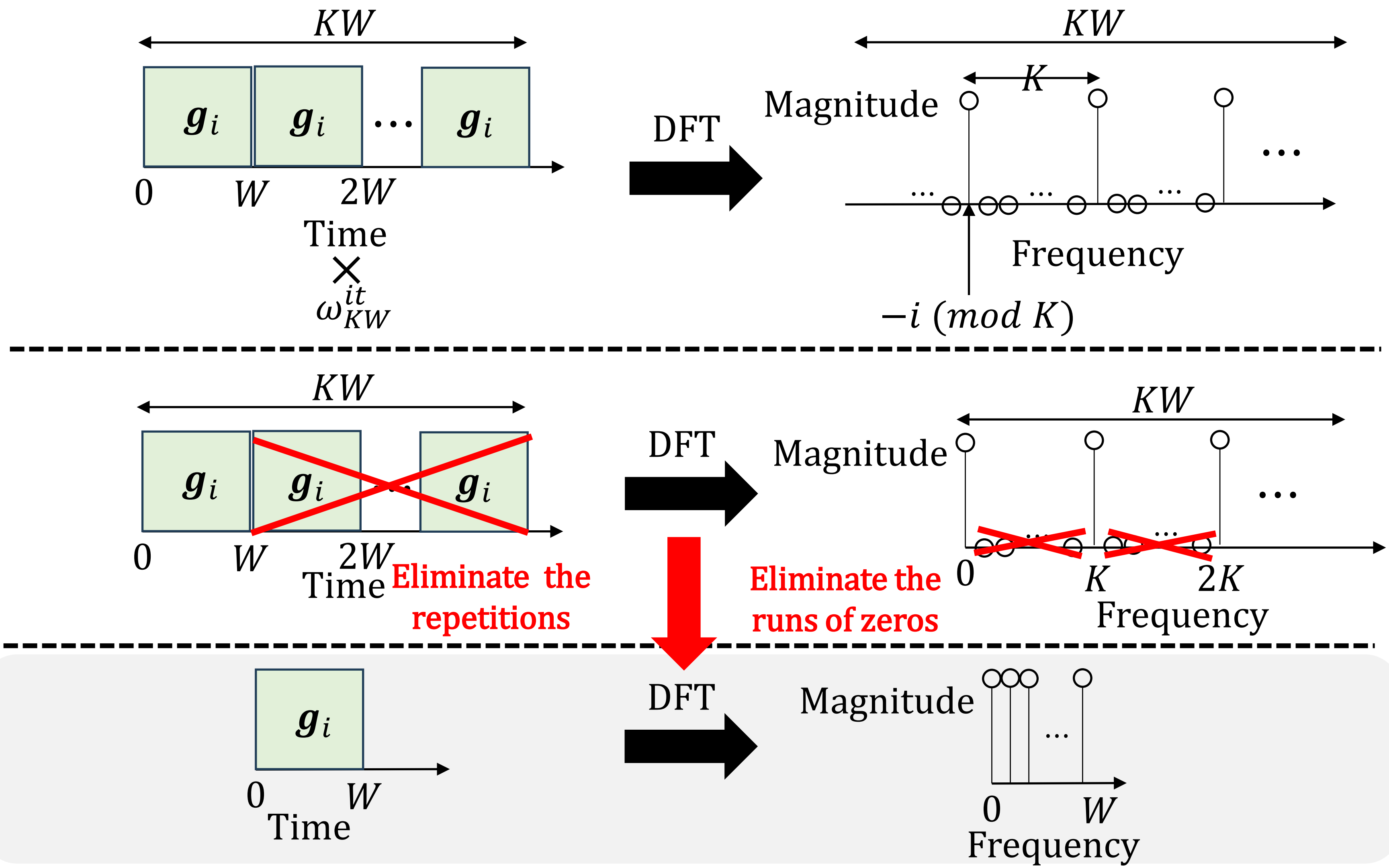
Proof : g_i must be a perfect sequence (3/5)

$$r_i(t) = g_i\big(t \pmod W\big)\omega_{KW}^{it}$$



Proof : g_i must be a perfect sequence (4/5)

$$r_i(t) = g_i\big(t \ (mod\ W)\big)\omega_{KW}^{it}$$



Proof : g_i must be a perfect sequence (5/5)

$\boldsymbol{g}_i$ has constant magnitude in the frequency domain.

$\Longrightarrow \boldsymbol{g}_i$ is a perfect sequence. $\blacksquare$

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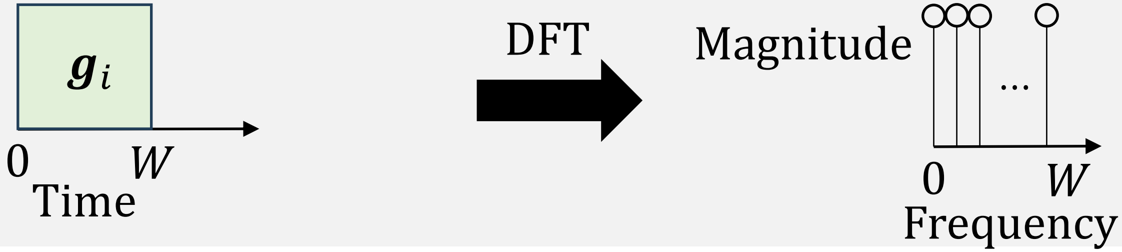
∴ Popović's construction describes all the uncorrelated optimal ZCZ sequence family.



Concluding Remarks

- **Within the category of ZCZ sequence families, there are optimal families that achieve the Tang-Fan-Matsufuji bound.**
- When addressing interference signals between users in multiple-access scenarios, the ideal scenario is when these optimal ZCZ sequence families satisfy the uncorrelatedness property
- Notably, Popović's construction generates all such uncorrelated optimal ZCZ sequence families, as demonstrated in our proof.

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