Crosscorrelation of q-ary Power Residue Sequences of Period p

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Example

- Ternary PRS (p=13, q=3, μ=2)

n	1	2	3	4	5	6	7	8	9	10	11	12
n^3	1	8	1	12	8	8	5	5	1	12	5	12

  - $C_0 = 2^0 \cdot C_0$ = {1,5,8,12}
  - $C_1 = 2^1 \cdot C_0$ = {2,10,3,11}
  - $C_2 = 2^2 \cdot C_0$ = {4,7,6,9}

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$S(n)$	0	0	1	1	2	0	2	2	0	2	1	1	0

Definition

- p : an odd prime
- q : a divisor of p-1
- μ : a primitive root mod p
- Coset Partition
  - C_0 : a set of q-th power residues mod p
  - $C_i = \mu^i \cdot C_0 \quad \text{for} \quad 0 \le i \le q-1$

◆ A q-ary PRS is defined as, for n = 0, 1, 2,…, p-1,

$$s(n) = \begin{cases} 0 & \text{if} \quad n \equiv 0 \pmod{p} \\ i & \text{if} \quad n \in C_i \ \text{ for } \ i \in Z_q \end{cases}$$

Some Properties (Known)

- Lemma 1 ('69, Sidel'nikov)
  - $\{s(n)\}$: q-ary PRS of period p
  - w : a primitive q-th root of unity
    - $s(1)=0$
    - For $u\neq 0$, $v\neq 0$ (mod p),
      $s(u)+s(v)\equiv s(uv)$
      $s(u)-s(v) \equiv s(u/v)$
    - For any $u\neq 0$ (mod p)

$$w^{s(-u)} = \begin{cases} -w^{s(u)}, & \text{if} \quad p \equiv q+1 \pmod{2q} \\ w^{s(u)}, & \text{if} \quad p \equiv 1 \pmod{2q} \end{cases}$$

    - $\sum_{n=1}^{p-1} w^{s(n)} = 0$

- When p=13, q=3, μ=2
  - $s(1)=0$
  - $s(2)+s(4)=1+2\equiv 0=s(8)$
    $s(3)-s(7)=1-2\equiv 2=s(3/7)=s(6)$
  - Since 13≡1 (mod 6),
    $s(1)=s(12)=0$, $s(2)=s(11)=1$
    etc.
  - $\sum_{n=1}^{12} w^{s(n)} = 4(w^0 + w^1 + w^2) = 0$

Autocorrelation

- Theorem 1 ('69, Sidel'nikov)
  - $\{s(n)\}$: a q-ary PRS of period p
  - w : a complex q-th root of unity
  - Autocorrelation of a q-ary PRS $\{s(n)\}$

$$R_s(\tau) = \sum_{x=0}^{p-1} w^{s(x+\tau)-s(x)} = \begin{cases} -1 - j2\beta(\tau) & \text{if } p \equiv q+1 \pmod{2q} \\ -1 + 2\alpha(\tau) & \text{if } p \equiv 1 \pmod{2q} \end{cases}$$

where $\alpha(\tau)$ and $\beta(\tau)$ are the real and imaginary part of $w^{s(\tau)}$.

$$\Rightarrow \qquad |R_s(\tau)| \le 3$$

How many distinct q-ary PRS ?

- change the primitive root mod p
- example (p=13, q=3)
  - using $\mu = 2, 6, 7, 11 = 2^1, 2^5, 2^{11}, 2^7$

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$s_1(n)$	0	0	1	1	2	0	2	2	0	2	1	1	0
$s_5(n)$	0	0	2	2	1	0	1	1	0	1	2	2	0
$s_{11}(n)$	0	0	2	2	1	0	1	1	0	1	2	2	0
$s_7(n)$	0	0	1	1	2	0	2	2	0	2	1	1	0

  - They are not all distinct!

Answer - Characterization

- Theorem 2
  - μ : a primitive root mod p
  - $\{s_i(n)\}$ and $\{s_j(n)\}$: q-ary PRS using μ^i and μ^j, respectively

  THEN $\exists v$ (mod q) such that $s_i(n) \equiv v \cdot s_j(n)$ (mod q), $\forall n$,

  where v is a solution to $j \equiv iv$ (mod p-1).

- Theorem 3
  - Denote by U_n the multiplicative group of integers mod n
  - p = an odd prime, q = a divisor of p-1

  THEN $U_{p\text{-}1} \equiv U_q$ (mod q)

- Corollary 1
  - The number of all the distinct q-ary PRS of period p is $\Phi(q)$.

Crosscorrelation of q-ary PRS

- Crosscorrelation between two q-ary sequences $\{s_1(n)\}$ and $\{s_2(n)\}$ of period p

$$C_{s_1,s_2}(\tau) = \sum_{n=0}^{p-1} w^{s_1(n+\tau)-s_2(n)}$$

- Theorem 4

  Two distinct q-ary PRS $\{s_1(n)\}$ and $\{s_2(n)\}$ of period p have:

$$\left|C_{s_1,s_2}(\tau)\right| \le \sqrt{p} + 2$$

Now, we have a sequence set with

- $\Phi(q)$ distinct PRS's of period p
- autocorrelation ≤ 3
- crosscorrelation $\le \sqrt{p}+2$

Interesting Observation

- Corollary 2 (2-level crosscorrelation)
  - p : an odd prime
  - q : a divisor of p-1
  - When $p \equiv q+1 \pmod{2q}$ and q even, consider a q-ary PRS $\{s(n)\}$.

For two distinct PRS $\{k_1 s(n)\}$ and $\{k_2 s(n)\}$ with $k_1 + k_2 \equiv 0 \pmod{q}$,

$$\left|C_{k_1 s, k_2 s}(\tau)\right| = \begin{cases} 1, & \tau = 0 \\ \sqrt{p}, & \tau \neq 0 \end{cases}$$

How good ?

- Welch's bound('74, Welch)
  - Periodic correlation bound of a signal set **S** with M signals of length L

$$C_{\max} \geq \sqrt{\frac{L^2(M-1)}{ML-1}}$$

  - $C_{\max} = \max\{R_{\max}^{\mathbf{a}}, C_{\max}^{\mathbf{a},\mathbf{b}} \mid \mathbf{a},\mathbf{b}(\neq \mathbf{a}) \in \mathbf{S}\}$

- Comparison with Welch's bound in the PRS's set

$$\sqrt{p} + 2 = C_{\max} \geq \sqrt{\frac{p^2(\phi(q)-1)}{p\phi(q)-1}} \approx \sqrt{p} \qquad (@\ p >> 1,\ \phi(q) >> 1)$$

Decimation of q-ary PRS

- Example (p=13, q=3, μ = 2)
  - Ternary sequences obtained by d-decimation of PRS of period 13

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$s(dn)\|_{d\in C_0}$	0	0	1	1	2	0	2	2	0	2	1	1	0
$s(dn)\|_{d\in C_1}$	0	1	2	2	0	1	0	0	1	0	2	2	1
$s(dn)\|_{d\in C_2}$	0	2	0	0	1	2	1	1	2	1	0	0	2

  - If $d_i, d_j \in C_k$, then $s(d_i n) = s(d_j n), \quad \forall n$
  - If $d_i \in C_k,\ d_j \in C_{l(\neq k)}$, then $s(d_i n) - s(d_j n) = k - l, \quad \forall n(\neq 0)$
  - The number of all the distinct decimations of a q-ary PRS is q.

    (They are not in general PRS.)

Crosscorrelation of two different decimation groups – VERY GOOD

- Theorem 5
  - $\{s(n)\}$: a q-ary PRS of period p
  - $D(s(n))$: set of all the decimations of $\{s(n)\}$
  - $D(ks(n))$: set of all the decimations of $\{ks(n)\}$
  - Crosscorrelation of

    any $\{s_1(n)\}=\{s(d_1 n)\} \in D(s(n))$ and

    any $\{s_2(n)\}=\{ks(d_2 n)\} \in D(ks(n))$:

$$\left| C_{s_1,s_2}(\tau) \right| \leq \sqrt{p} + 2$$

Crosscorrelation inside a single decimation group – NOT GOOD

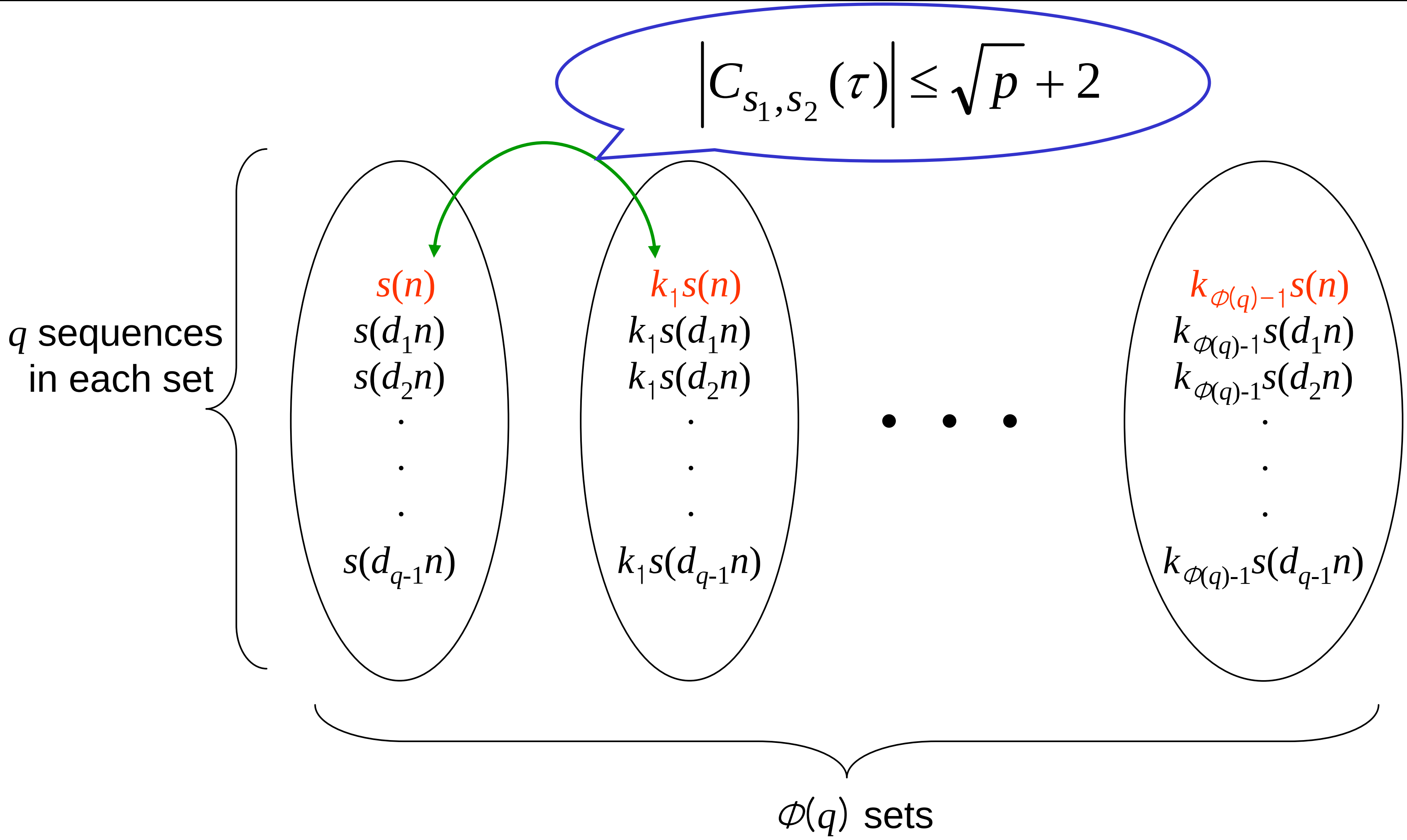
- Theorem 6
  - $\{s(n)\}$: a q-ary PRS of period p
  - $D(s(n))$: a set of all decimations of $\{s(n)\}$
  - Crosscorrelation of any two distinct q-ary sequences $\{s_1(n)\}=\{s(d_1 n)\} \in D(s(n))$ and $\{s_2(n)\}=\{s(d_2 n)\} \in D(s(n))$ of period p

$$C_{s_1,s_2}(\tau) = \begin{cases} 1+(p-1)w^{-m}, & \text{if } \tau = 0 \\ w^{s(\tau)}[1-w^{-m}]+w^{-m}R_s(\tau), & \text{if } \tau \neq 0 \end{cases}$$

where m is an integer satisfying $\frac{d_2}{d_1} \in C_{m\neq 0}$.

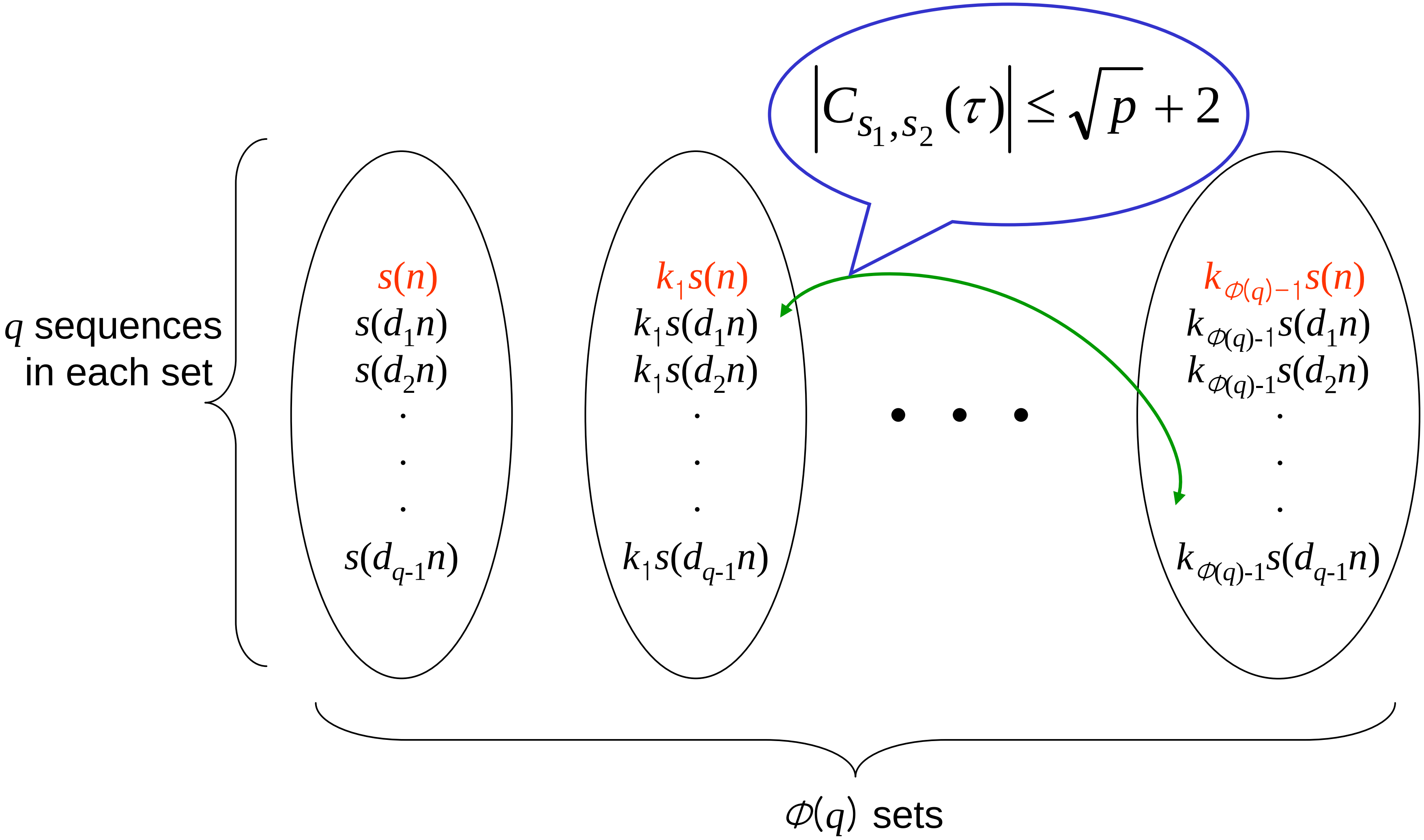
$$\Rightarrow \quad \left|C_{s_1,s_2}(\tau)\right| = \begin{cases} \approx p, & \tau = 0 \\ \leq 5, & \tau \neq 0 \end{cases}$$

Big Picture



$\left|C_{s_1,s_2}(\tau)\right| \le \sqrt{p}+2$
q sequences in each set
$s(n)$
$s(d_1 n)$
$s(d_2 n)$
$s(d_{q-1} n)$
$k_1 s(n)$
$k_1 s(d_1 n)$
$k_1 s(d_2 n)$
$k_1 s(d_{q-1} n)$
$k_{\Phi(q)-1} s(n)$
$k_{\Phi(q)-1} s(d_1 n)$
$k_{\Phi(q)-1} s(d_2 n)$
$k_{\Phi(q)-1} s(d_{q-1} n)$
$\Phi(q)$ sets

Big Picture


$\left|C_{s_1,s_2}(\tau)\right| \le \sqrt{p} + 2$
q sequences in each set
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$s(d_1 n)$
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$k_1 s(d_2 n)$
$k_1 s(d_{q-1} n)$
$k_{\Phi(q)-1} s(n)$
$k_{\Phi(q)-1} s(d_1 n)$
$k_{\Phi(q)-1} s(d_2 n)$
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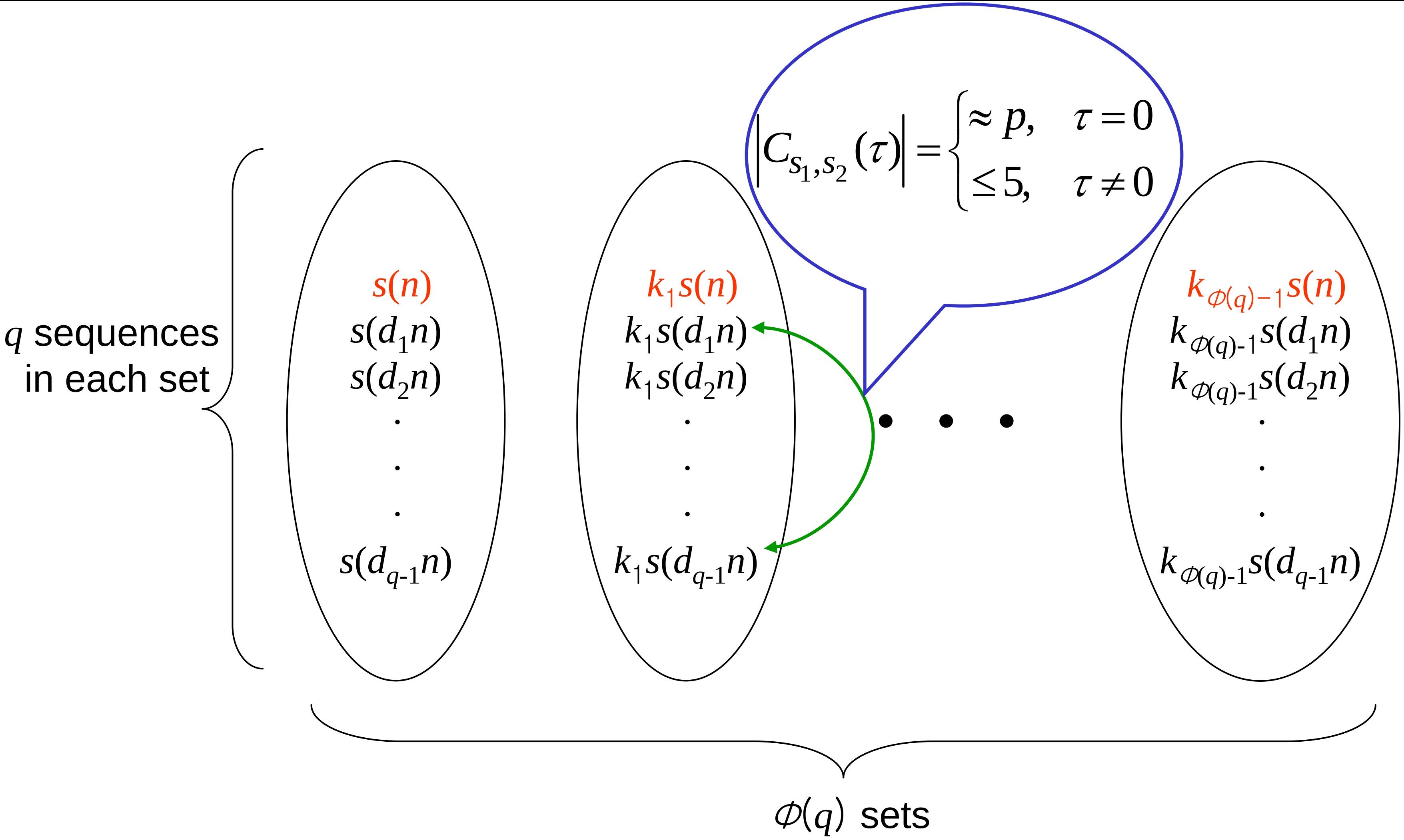
$\Phi(q)$ sets

Big Picture


$|C_{s_1,s_2}(\tau)| = \begin{cases} \approx p, & \tau = 0 \\ \leq 5, & \tau \neq 0 \end{cases}$
q sequences in each set
$s(n)$
$s(d_1 n)$
$s(d_2 n)$
$s(d_{q-1} n)$
$k_1 s(n)$
$k_1 s(d_1 n)$
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$k_{\Phi(q)-1} s(d_{q-1} n)$
$\Phi(q)$ sets

Comments

- There is another type of q-ary sequence $\{t(n)\}$ of length p^m-1, known as Sidel'nikov sequence (or Lempel-Cohn-Eastman sequence), where q is a divisor of p^m-1 and m is a positive integer.

- (IT-submitted)

  Crosscorrelation of a set which consists of q-ary Sidel'nikov sequence $\{t(n)\}$ of length p^m-1 and its constant multiple sequences

$$\left|C_{t_1,t_2}(\tau)\right| \le \sqrt{p^m} + 3$$