A Probabilistic Approach on Estimating the Number of Modular Sonar Sequences

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In this talk

- A Brief Review about Sonar Sequences
- Initial Exhaustive Search
- Probabilistic Model
- Comparing Models with True Value
- Concluding Remarks

A Brief Review about Sonar Sequences

What is Sonar Sequence?

m=9										
8			•		•					
7	•									
6		•								
5				•			•			
4										
3								•	•	
2										
1										
0						•				
	1	2	3	4	5	6	7	8	9	n=9

- Can we select one dot for all columns to make no distinct two dots make same line segment?
- Upper picture is it-The integer sequence becomes: 7 6 8 5 8 0 5 3 3

A Brief Review about Sonar Sequences

Definition of Sonar Sequence and Modular Sonar Sequence

A function $f : A_n (\triangleq \{1, 2, ..., n\}) \rightarrow A_m$ has a distinct difference property (DDP) if for all integers h, i, and j, with $1 \leq h \leq n-1$ and $1 \leq i, j \leq n-h$,

$$f(i+h) - f(i) = f(j+h) - f(j) \quad \text{implies} \quad i = j. \tag{1}$$

And has a distinct modular difference property (DMDP) if

$$f(i+h) - f(i) = f(j+h) - f(j) \pmod{m} \quad \text{implies} \quad i = j. \tag{2}$$

Then a sequence with DDP is a Sonar Sequence, and it with DMDP is a Modular Sonar Sequence

A Brief Review about Sonar Sequences

Difference Triangle of Sonar Sequence

7		**6**		**8**		**5**		**8**		**0**		**5**		**3**		**3**
	-1		2		-3		3		-8		5		-2		0	
		1		-1		0		-5		-3		3		-2		
			-2		2		-8		0		-5		3			
				1		-6		-3		-2		-5				
					-7		-1		-5		-2					
						-2		-3		-5						
							-4		-3							
								-4								

- All row elements of each difference rows are distinct!

A Brief Review about Sonar Sequences

Difference Triangle of Modular Sonar Sequence

7		**6**		**8**		**5**		**8**		**0**		**5**		**3**		**3**
	8		2		6		3		1		5		7		0	
		1		8		0		4		6		3		7		
			7		2		1		0		4		3			
				1		3		6		7		4				
					2		8		4		7					
						7		6		4						
							5		6							
								5								

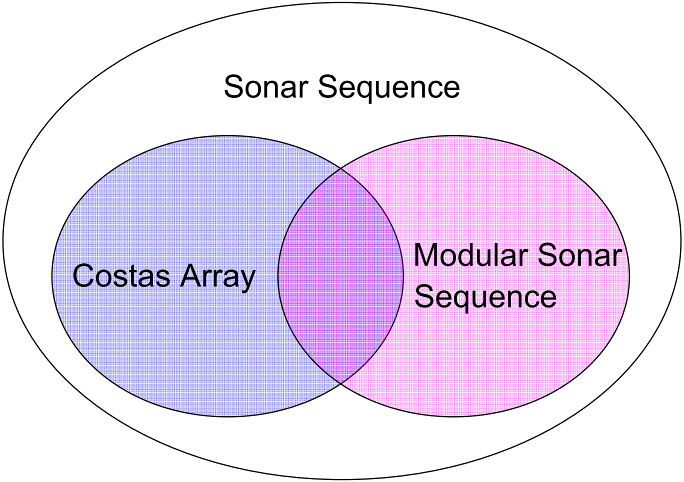
- All difference values are considered same when they are congruent by *mod* m at Modular Sonar Sequence. In this case, $m = 9$.

Why Modular Sonar Sequence?

- Usually, Sonar Sequences are for use in communication applications. Their optimum (maximum) length is $2m$, but it's hard to find good (long) sequences when m is large.
- Modular Sonar Sequences have some powerful construction methods that can reach their optimum length, $m+1$.
- They have some interesting and useful property, and are easy to analyze. Above all, they are easily converted and expanded to Sonar Sequences.

A Brief Review about Sonar Sequences

Relations between Sonar, Modular Sonar Sequences and Costas Arrays



A Brief Review about Sonar Sequences

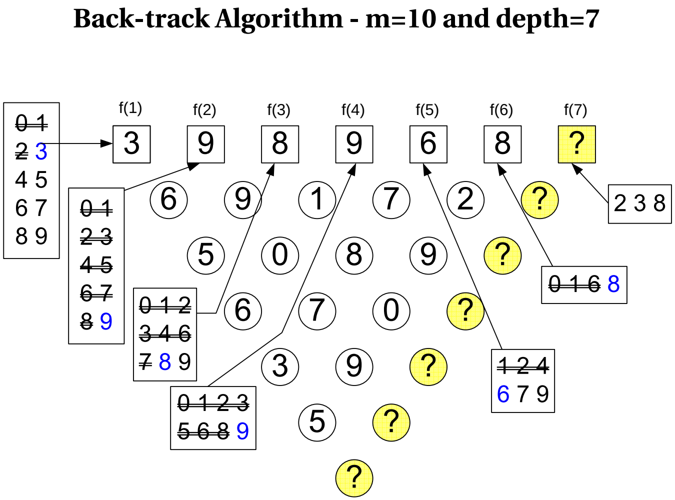
Some Properties and Algebraic Constructions of MSS

- Equivalant Sequences : If an f is a modular sonar sequence, the function g given by

$$g(i) = uf(i) + si + a, \quad i = 1, 2, \ldots, n \tag{3}$$

  is also a modular sonar sequence for all integer s and a, and for all integer u relatively prime to m.
- Reduction : Given an $m \times n$ (modular) sonar sequence, we can always have $m \times (n-1)$ (modular) sonar sequence by deleting the last term.
- Algebraic Constructions for $m \times (m+1)$ exist for $m = p$ and $m = p^k - 1$ for any prime p and a positive integer k.
  1. p : Quadratic Method, Extended Exponential Welch Method*
  2. $p^k - 1$: Shift Sequence Method

Initial Exhaustive Search



Initial Exhaustive Search

Object of Search

- The initial search was to answer the following two questions:

Q.1 Determine the maximum length n_{max} such that an $m \times n_{max}$ modular sonar array exists. What would be the maximum length n_e if we count only those that are NOT equivalent to any examples constructed by the three algebraic methods mentioned in the previous section and/or their reductions?

Q.2 Determine the number $I(m)$ of inequivalent sequences of length n_e for a given m, excluding those which are equivalent to the one given by the three algebraic constructions and/or their reductions.

Initial Exhaustive Search

Result of an Initial Search (part)

Description	m	n_{max}	n_e	$m-n_e$	$I(m)$
U	9	10	10	-1	3
SS	10	11			
Q,EEW	11	12	11	0	30
SS	12	13			
Q,EEW	13	14	13	0	17
U	14	14	14	0	2
SS	15	16	15	0	1
SS	16	17	16	0	1
Q,EEW	17	18	16	1	33
SS	18	19			
Q,EEW	19	20	17	2	321
U	20	18	18	2	136
U	21	19	19	2	17
SS	22	23			
Q,EEW	23	24			
SS	24	25			
U	25	22	22	3	4

SS:Shift Sequence **Q**:Quadratic **EEW**:Extended Exponential Welch **U**:Unidentified

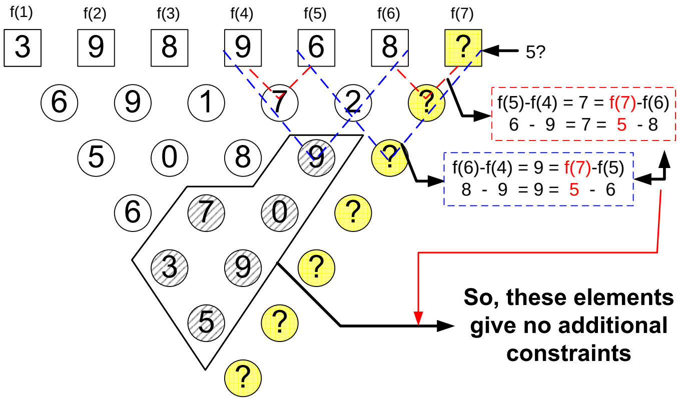
Initial Exhaustive Search

Two Observations of the Result

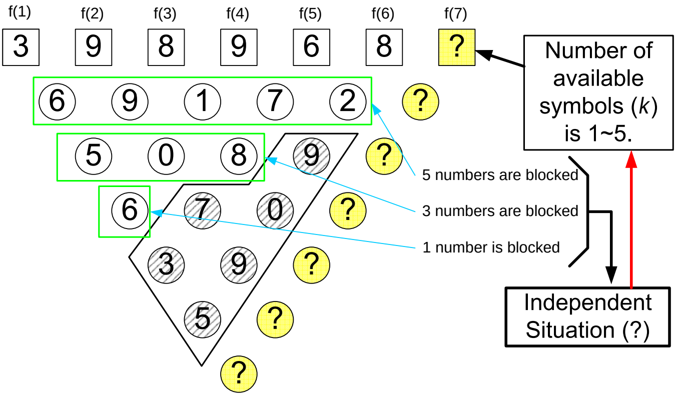
- $m - n_e$ is monotonically non-decreasing as m is increasing.
- $I(m)$ is decreasing as m is increasing for the range where the value $m - n_e$ remains the same.

Probabilistic Model

Back-track Algorithm - Remind



Probabilistic Model

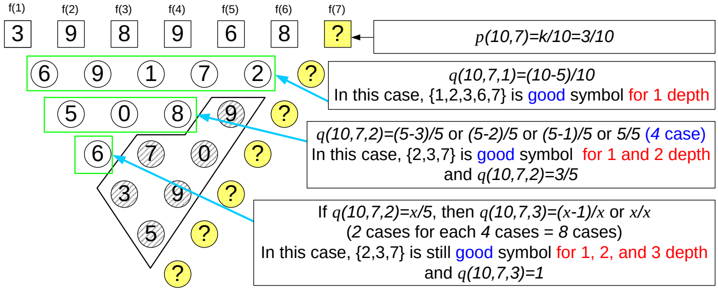


- We would like to estimate the number k even we do not know former terms.
- If the number of all 10×6 MSS is w, we could say the number of all 10×7 MSS is wk (?).

Probabilistic Model

Some Notations and Analysis

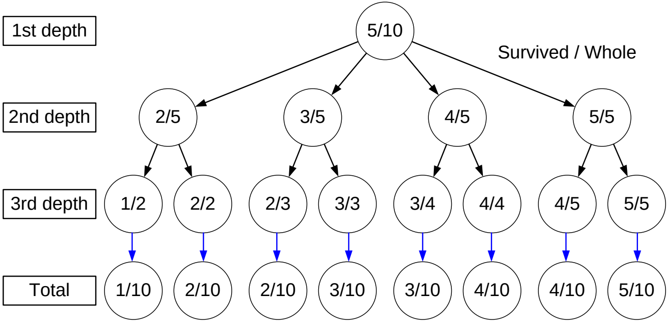
- $N(m,t)$ =The expected number of $m \times t$ Modular Sonar Sequences.
- $p(m,t)$ =The fraction of number of t-th symbol that give distinct value for whole difference rows.
- $q(m,t,h)$ =The fraction of number of t-th symbol that give distinct value for h-th difference row. m is modulo value for difference.



$p(10,7)=3/10=q(10,7,1)\times q(10,7,2)\times q(10,7,3)=5/10\times 3/5\times 1$

Probabilistic Model

General Case: m=10, t=7



- Total fraction is derived by the product of each fractions of depth on their path.
- Can we find one value for a row that can replace each cases with minimal error? In general m and t, can we find equation of m, t that estimates the true value well?

Probabilistic Model

Our Claim - Main Result

$$q(m,t,h) \approx 1 - \frac{m(t-2h)}{(m-h+1)^2}, \quad \text{for } 1 \le h \le \lfloor \tfrac{t}{2} \rfloor. \tag{4}$$

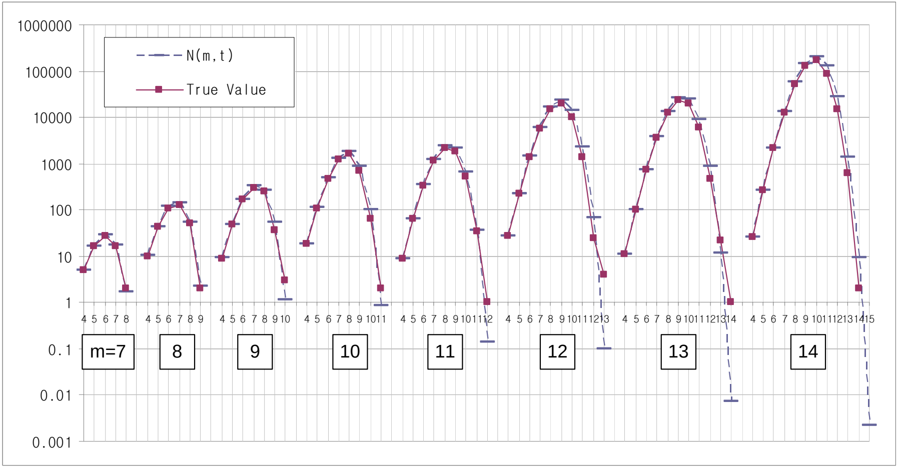
- And, since $p(m,t) \approx \prod_{h=1}^{\lfloor \frac{t}{2} \rfloor} q(m,t,h)$ and $N(m,n) \approx N(m,n-1)\,mp(m,n)$ by our assumptions,

$$\begin{aligned} N(m,1) &= m, \\ N(m,n) &\approx m^n \prod_{t=1}^{n} \prod_{h=1}^{\lfloor \frac{n}{2} \rfloor} \left(1 - \frac{m(t-2h)}{(m-h+1)^2}\right), \quad 2 \le n \le m+1. \end{aligned} \tag{5}$$

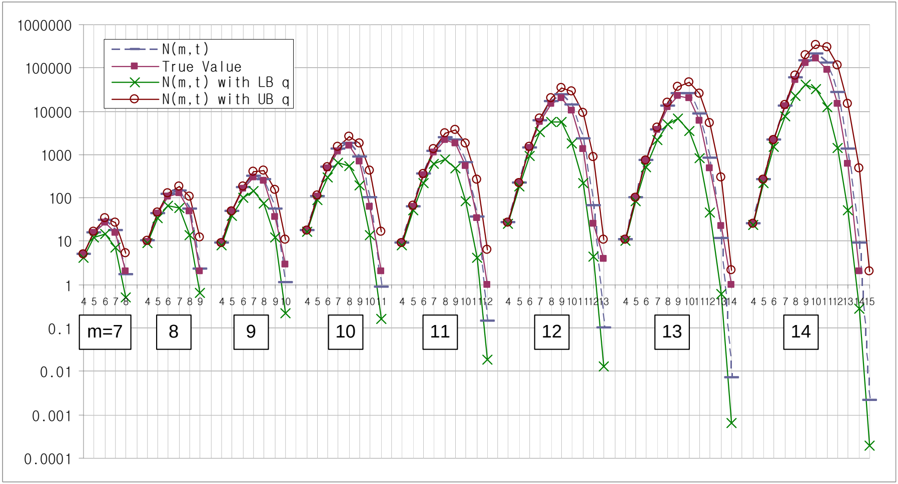
Comparing Models with True Value

m	n	TV	$N(m,n)$	m	n	TV	$N(m,n)$	m	n	TV	$N(m,n)$
7	4	5	5.00	10	9	707	895	13	5	100	100
	5	16	16.1		10	63	103		6	729	738
	6	27	29.5		11	2	0.857		7	3712	3842
	7	16	17.7	11	4	9	9.00		8	12433	13492
	8	2	1.73		5	64	64.1		9	22983	26184
8	4	10	10.5		6	343	350		10	20198	25321
	5	43	43.9		7	1152	1215		11	5922	8835
	6	108	118		8	2209	2479		12	481	852
	7	128	140		9	1857	2190		13	22	11.5
	8	50	54.3		10	533	670		14	1	0.0073
	9	2	2.26		11	35	37.1	14	4	26	26.0
9	4	9	9.33		12	1	0.142		5	262	262
	5	48	48.1	12	4	27	27.5		6	2160	2188
	6	167	173		5	222	223		7	12896	13362
	7	292	326		6	1399	1430		8	53373	57579
	8	249	271		7	5848	6187		9	130547	147892
	9	37	54.2		8	15324	17022		10	168576	209626
	10	3	1.12		9	20155	23392		11	87718	127056
10	4	18	18.0		10	10199	13865		12	14775	27624
	5	110	110		11	1351	2297		13	615	1362
	6	480	499		12	25	67.7		14	2	9.14
	7	1216	1325		13	4	0.0996		15	0	0.0022
	8	1619	1845	13	4	11	11.0				

Comparing Models with True Value

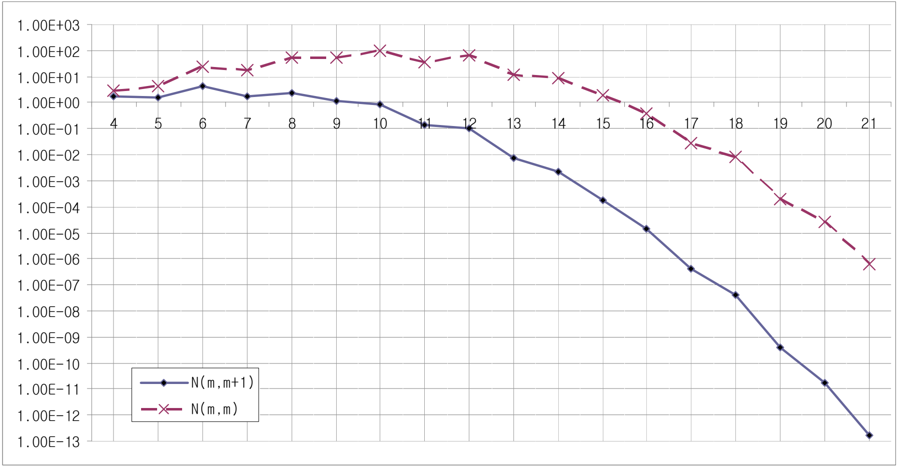


Comparing Models with True Value



Comparing Models with True Value

Why Decaying?



Concluding Remarks

- We have checked the existence of $m \times n$ modular sonar sequences by computer search for some small values of m
- We estimated the number of inequivalent examples for various values of m by probabilistic approach.
- From this estimate, we could have concluded that no full-size modular sonar sequence exists for m beyond a certain value. This is, however, not true.
- We could safely guess that any full-size example for large values of m must be either from an algebraic construction, or else the probability that it exists is extremely small.

Concluding Remarks

Some Unsolved Problems

1. Find an example of 35×35 modular sonar sequences (mod 35) or prove that none exists.
2. Generalize the above to the case of $m = p(p+2)$ being a product of twin primes.
3. Find infinitely many values of m for which an $m \times (m+1)$ modular sonar sequences do not exist.
4. Except for m being a prime or one less than a prime power, would the fact that the value in (5) is close to zero imply non-existence?
5. How accurate is the estimate in (5)?
6. Could a similar approach be used to estimate the number of Costas arrays?

Concluding Remarks

Thank You!

Any Questions?