Generalized Locality for Distributed Storage Codes

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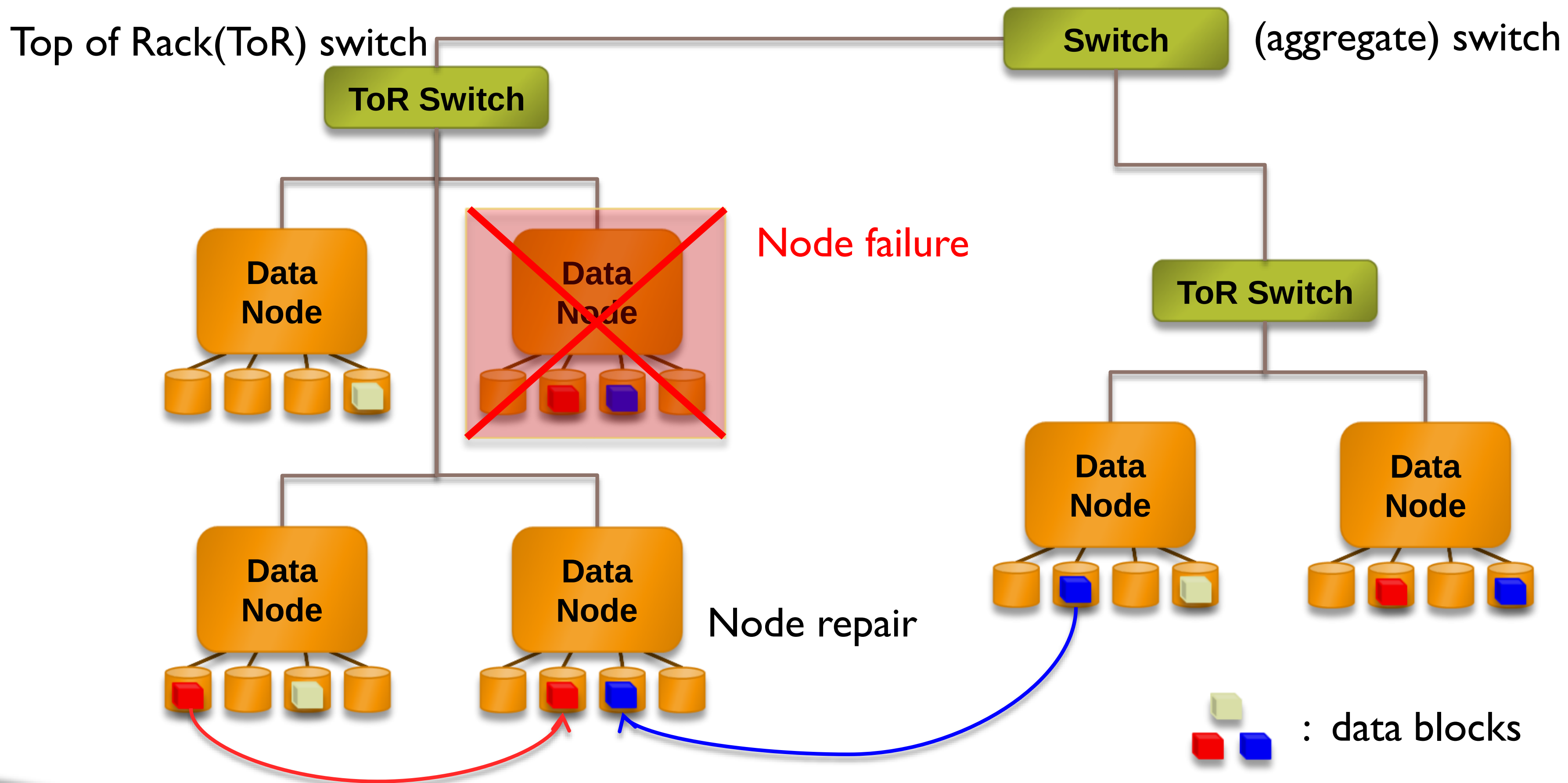
Distributed Storage Codes

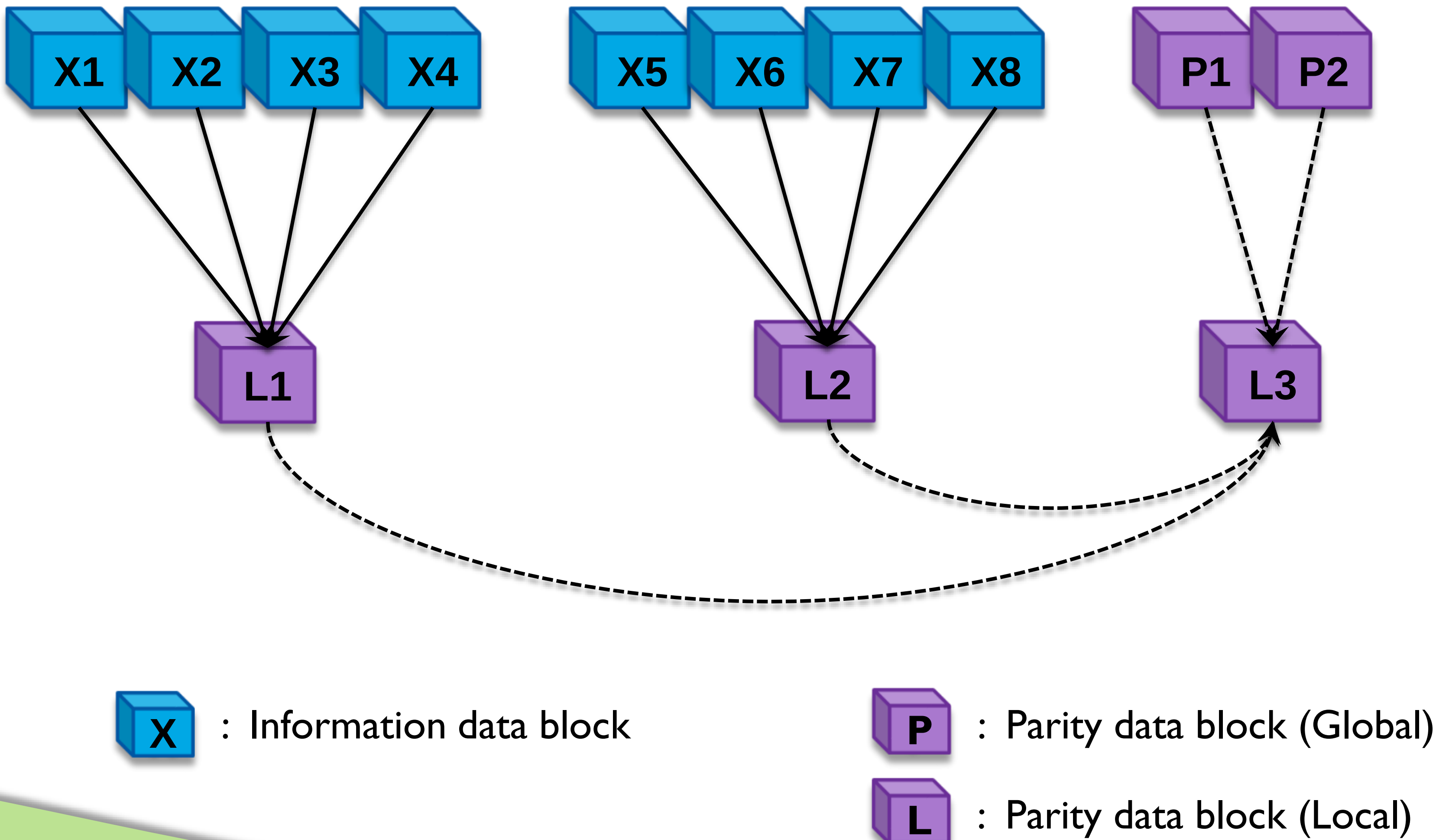
- Codes for Distributed Storage Systems
  - Store data while protecting against node failures



Switch : transfer data blocks **Data Node** : store data blocks

Locally Repairable Codes (LRC)

- How to reduce the cost for data repair?



Concept of Locality

(Gopalan-12) introduced codes with locality & minimum distance bound

(Papailiopoulos-12) generalized the minimum distance bound for non-linear codes and vector codes

(Tamo-14, Prakash-14, Rawat-14) devoted the locality in handling of multiple node failures

A. S. Rawat, A. Mazumdar, and S. Vishwanath. (2014). "Cooperative local repair in distributed storage," [Online]. Available: http://arxiv.org/abs/1409.3900.

(This presentation) introduces generalized locality and related bounds

Construction of Codes with Locality

(Huang-07) Pyramid codes

(Kamath-13) MSR/MBR-local codes

(Huang-12) Local reconstruction codes

(Sathiamoorthy-13) Locally repairable codes

(Krishnan-14, Shahabinejad-14) Codes with locality for Hadoop

(This presentation) proposes a construction of codes with GOOD generalized locality

Generalized Locality (GL)

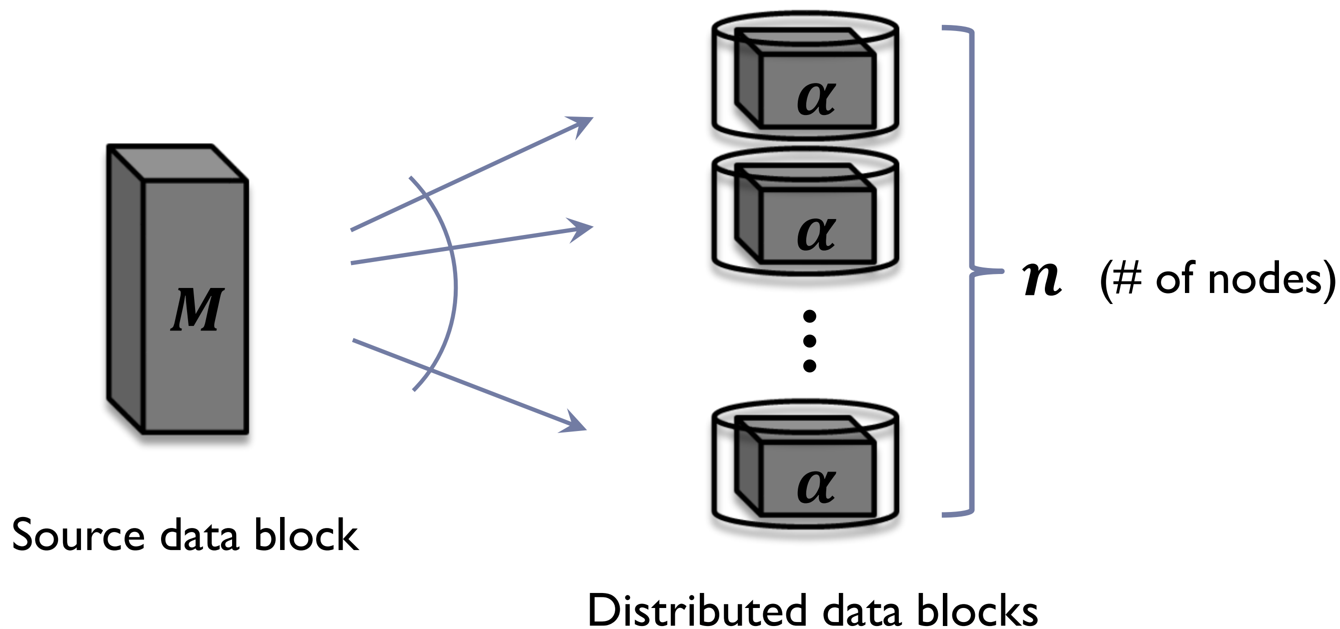
- Preliminary

Let $G: \mathbb{F}_q^M \mapsto \mathbb{F}_q^{n\cdot\alpha}$ be an encoding (generator) function:

$$G(\boldsymbol{X}) = \boldsymbol{Y} = (Y_1, Y_2, \dots, Y_n),$$

where $\boldsymbol{X} \in \mathbb{F}_q^M$ and $\boldsymbol{Y} \in \mathbb{F}_q^{n\cdot\alpha}$.

We denote the code determined by the encoding function G as an $(n, (M, \alpha), d)_q$ code $\mathcal{C}$, where d is the minimum distance.

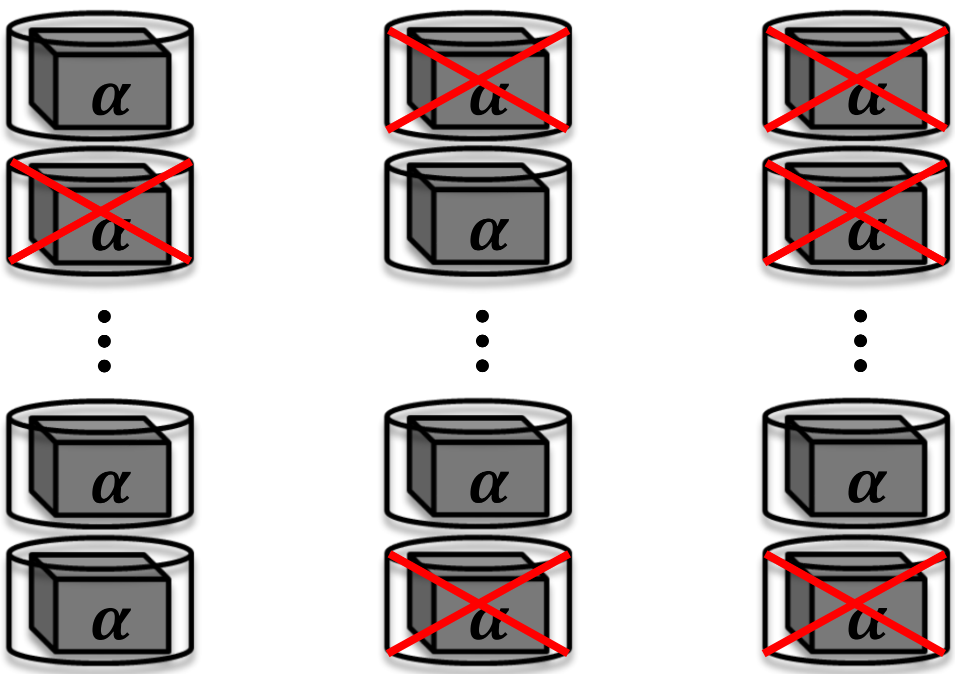


Generalized Locality (GL)

- Motivation

  To construct distributed storage codes,

  We should consider **all possible failure patterns** **together**.

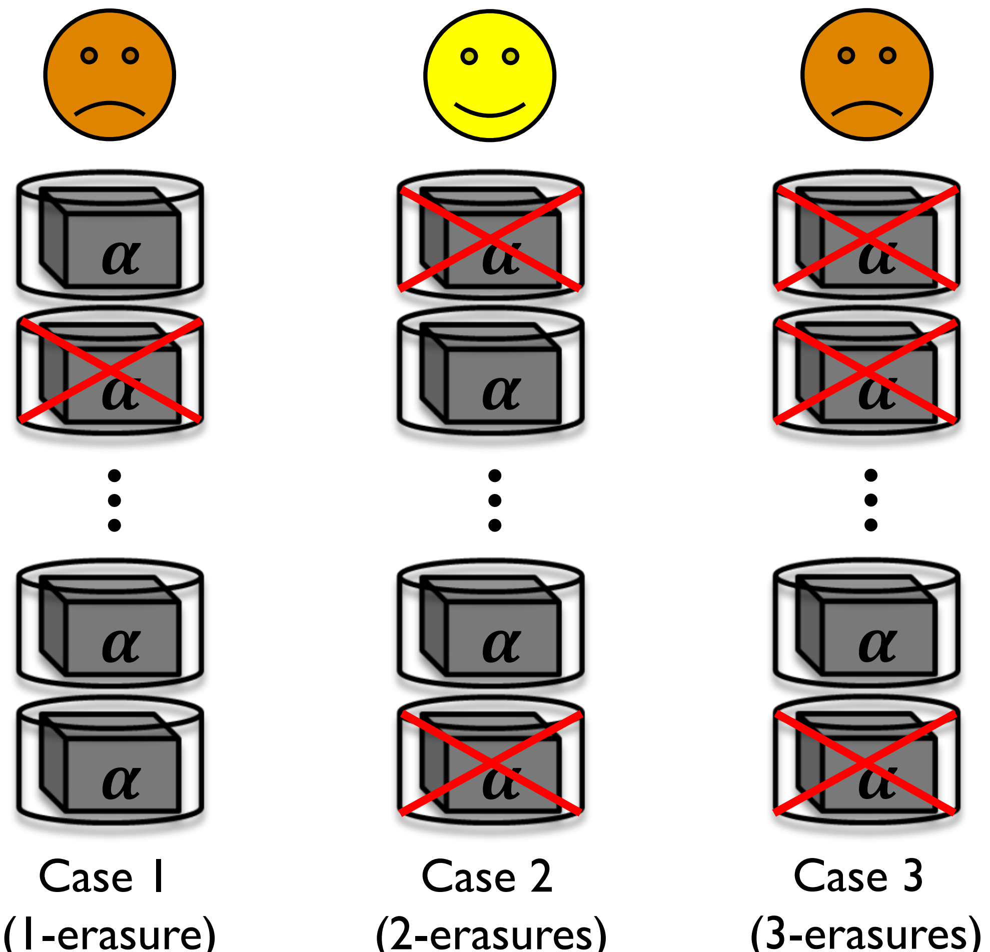


Case 1 (1-erasure) | Case 2 (2-erasures) | Case 3 (3-erasures)

Generalized Locality (GL)

- Motivation

Assume that the code is optimized **only** for **2-erasures**



Generalized Locality (GL)

Definition

Consider an $(n, (M, \alpha), d)_q$ code $\mathcal{C}$ given by the above. Let ℓ be an integer with $1 \le \ell \le d-1$, and $E \subset [n]$ with $|E| = \ell$. We denote by $\boldsymbol{Y}_E$ the set of coded symbols $Y_i, i \in E$. That is $\boldsymbol{Y}_E = \{Y_i | i \in E, |E| = \ell\}$. Then,

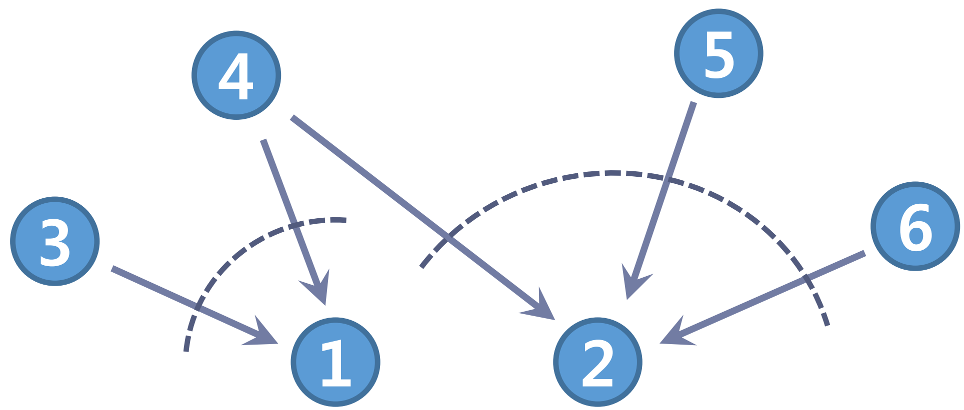
1) $R(E) \subseteq [n] \setminus E$ is called a repair set for $\boldsymbol{Y}_E$ if every $Y_i, i \in E$ can be regenerated by a set of functions on $Y_j, j \in R(E)$. **(Rawat-14)**
Note that there can be many different repair sets for a given set of symbols $\boldsymbol{Y}_E$.

2) The integer r is called the locality of $\boldsymbol{Y}_E$ if r is the minimum of all the cardinalities of the repair sets for $\boldsymbol{Y}_E$.
It is called the ℓ-locality of $\mathcal{C}$ if every set of symbols $\boldsymbol{Y}_E, |E| = \ell$, has the locality at most r. The ℓ-locality of $\mathcal{C}$ is denoted by r_ℓ. **(Rawat-14)**

3) The set of integers, $r_1, r_2, \ldots, r_{d-1}$, is called the generalized locality of $\mathcal{C}$.

Generalized Locality (GL)

- Example 1



A repair set of Y_1 : $R(1) = \{3,4\}$
A repair set of Y_2 : $R(2) = \{4,5,6\}$

Locality of Y_1 : 2
Locality of Y_2 : 3

Let $E = \{1,2\}$.
A repair set of $\boldsymbol{Y_E}$: $R(E) = \{3,4,5,6\}$

Locality of Y_E : 4

Locality (1-locality) of $\mathcal{C}$: 3
2-locality of $\mathcal{C}$: 4
Generalized Locality of $\boldsymbol{\mathcal{C}}$: $\boldsymbol{(r_1 = 3, r_2 = 4)}$

Bounds for Codes with GL

- Theorem 1

Let $\mathcal{C}$ be an $(n,(M,\alpha),d)_q$ code with generalized locality $(r_1, r_2, \dots, r_\ell,)$. Then, the minimum distance of $\mathcal{C}$ is bounded as

$$d \leq \min_{\ell \geq 1}\left(n - \left\lceil \frac{M}{\alpha} \right\rceil + 1 - \ell \cdot \left(\left\lceil \frac{M}{r_\ell \cdot \alpha} \right\rceil - 1\right)\right).$$

	Target Codes				Locality Types		
	Scalar ($\alpha = 1$)	Vector ($\alpha \geq 1$)	Linear	Non-linear	1-locality ($\ell = 1$)	ℓ-locality ($\ell \geq 1$)	Generalized locality
Gopalan-12	O		O		O		
Forbes-13	O		O	O	O		
Papailiopoulos-12	O	O	O	O	O		
Rawat-14	O		O	O	O	O	
Ours	**O**	**O**	**O**	**O**	**O**	**O**	**O**

Simpler version (linear and scalar) and some derived new bounds

Let $\mathcal{C}$ be an $[n,k,d]_q$ code with generalized locality $(r_1, r_2, \dots, r_\ell,)$.

$$d \le \min_{\ell \ge 1}\left(n-k+1-\ell\cdot\left(\left\lceil\frac{k}{r_\ell}\right\rceil-1\right)\right)$$

and, hence,

$$R(\mathcal{C}) \le \min_{\ell\ge 1}\left(\frac{r_\ell}{r_\ell+\ell}\right)$$

and

$$\left\lceil\frac{k\cdot\ell}{n-k+1-d+\ell}\right\rceil \le r_\ell \le k \quad \text{for } 1 \le \ell \le d-1$$

Example: For $[7,3,4]_2$ code and $\ell = 3$,

$$3 = \left\lceil\frac{3\cdot 3}{7-3+1-4+3}\right\rceil \le r_\ell \le 3 \text{ implies } r_3 = 3.$$

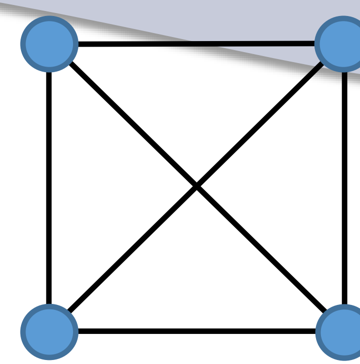
Construction of codes with $(r_1, r_2) = (2, 3)$

- One choice would be simplex codes with the parameter $n = 2^k - 1, k, d = 2^{k-1}$.
- Examples:
  - $G = \begin{pmatrix} 100 & 110 & 1 \\ 010 & 101 & 1 \\ 001 & 011 & 1 \end{pmatrix}$ for k=3
  - $G = \begin{pmatrix} 1000 & 111000 & 1110 & 1 \\ 0100 & 100110 & 1101 & 1 \\ 0010 & 010101 & 1011 & 1 \\ 0001 & 001011 & 0111 & 1 \end{pmatrix}$ for k=4

$r_\ell \leq \ell + 1$ (Rawat-14)

- It has $(r_1, r_2) = (2, \mathbf{3})$.
- Code rate $= \frac{k}{2^{k-1}}$ (VERY LOW)
- **Question**: Can we improve the rate maintaining the property $(\boldsymbol{r_1}, \boldsymbol{r_2}) = (\mathbf{2}, \mathbf{3})$?

One simple idea that works

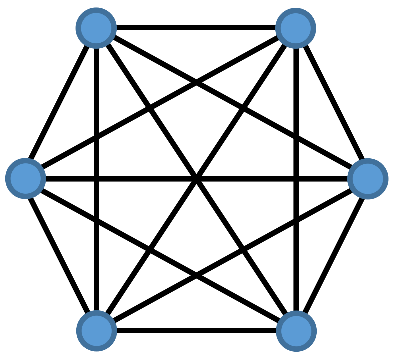


- $G = \begin{pmatrix} 1000 & 111000 & \cancel{1110} & \cancel{1} \\ 0100 & 100110 & \cancel{1101} & \cancel{1} \\ 0010 & 010101 & \cancel{1011} & \cancel{1} \\ 0001 & 001011 & \cancel{0111} & \cancel{1} \end{pmatrix}$ for k=4
- Can we prove that this modified code STILL has $(\boldsymbol{r_1}, \boldsymbol{r_2}) = (\mathbf{2}, \mathbf{3})$?
- What is the code ? Its generator matrix has all the columns of weight 1 and weight 2 ONLY.
- It can be described as a complete graph with 4 vertices.

  Columns are all the vertices and edges of K_4

Complete Graph Codes

- Construction of $[k(k+1)/2\,,k,\boldsymbol{k}]_2$ codes



K_6 **complete graph ($k=6$)**

$$\left(\begin{array}{cccccc:ccccccccccccccc}
1&0&0&0&0&0&1&1&0&1&0&0&1&0&0&0&1&0&0&0&0\\
0&1&0&0&0&0&1&0&1&0&1&0&0&1&0&0&0&1&0&0&0\\
0&0&1&0&0&0&0&1&1&0&0&1&0&0&1&0&0&0&1&0&0\\
0&0&0&1&0&0&0&0&0&1&1&1&0&0&0&1&0&0&0&1&0\\
0&0&0&0&1&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0&1\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1
\end{array}\right)$$

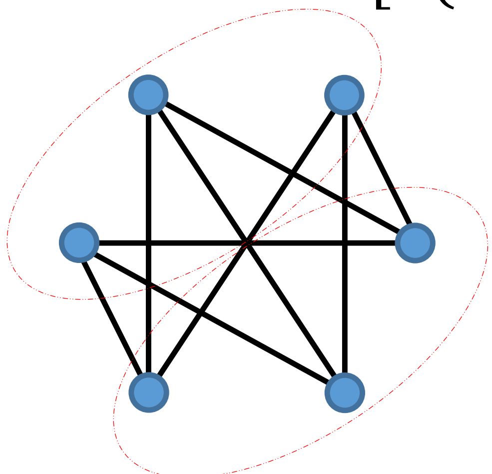
Generator matrix of $[21,6,6]_2$ code

Theorem: It has $d_{min} = k$ and

1) $r_1 = 2$ for $k \geq 2$.
2) $r_2 = 3$ for $k \geq 3$.
3) $r_\ell \leq \min(2\ell, k)$ for $k \geq 2$ and $\ell \in [k-1]$.

Complete Multipartite (p-partite) Graph Codes

- Construction of $[k(k-q+2)/2\,, k, k-q+1]_2$ codes



Complete 2−partite 3−uniform graph with 6 vertices ($p=2, q=3, k=6$)

$$\left(\begin{array}{cccccc|ccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1
\end{array}\right)$$

Generator matrix of $[15, 6, 4]_2$ code

Theorem: It has $d_{min} = k - q + 1$ and

1) $r_1 = 2$ for $k \geq 2$.

2) $r_2 = \begin{cases} 3, & \text{for } q = 1 \text{ and } k \geq 3, \\ 4, & \text{for } q \geq 2 \text{ and } k \geq 3. \end{cases}$

3) $r_\ell \leq \min(2\ell, k)$ for $k \geq 2$ and $\ell \in [k-q]$.

Complete Multipartite Graph Codes

- Code rates

$$R = \frac{2}{k-q+2} \quad \geq \quad R_S = \frac{k}{2^k - 1}$$

- Minimum distances

$$d = k - q + 1 \quad \leq d_S = 2^{k-1}$$

For complete multipartite graph codes with dimension k,

\# of partitions

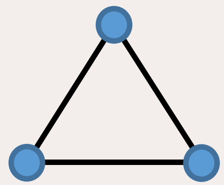
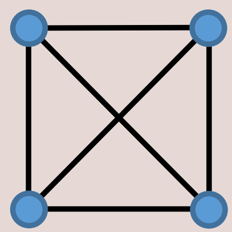
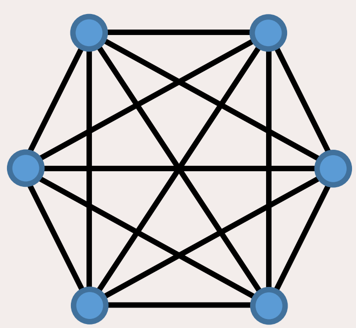
$q \uparrow$ or $p \downarrow$ $\Rightarrow$ $R \uparrow$ and $d \downarrow$

Size of a partition

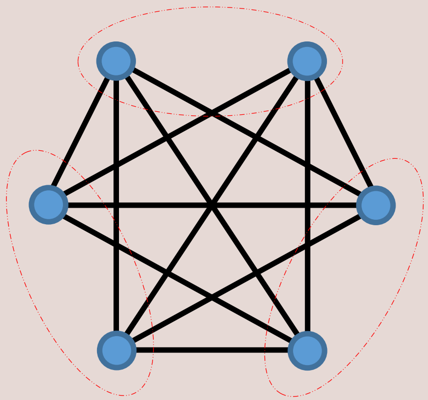
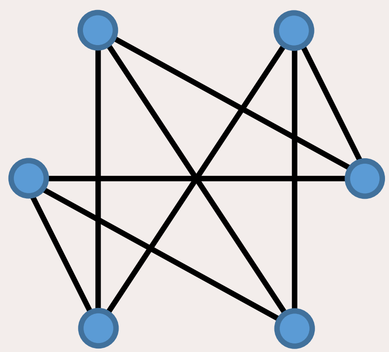
Code rate

Minimum distance

Complete Multipartite Graph Codes

Codes	Graphs	Generator matrices
$[3, 2, 2]_2$ **code** $(q = 1, k = 2)$	 (Complete graph)	$\begin{pmatrix} 1 & 0 & \vert & 1 \\ 0 & 1 & \vert & 1 \end{pmatrix}$
$[6, 3, 3]_2$ **code** $(q = 1, k = 3)$	 (Complete graph)	$\left(\begin{array}{ccc\|ccc} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right)$
$[10, 4, 4]_2$ **code** $(q = 1, k = 4)$	 (Complete graph)	$\left(\begin{array}{cccc\|cccccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right)$
$[21, 6, 6]_2$ **code** $(q = 1, k = 6)$	 (Complete graph)	$\left(\begin{array}{cccccc\|ccccccccccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{array}\right)$

Complete Multipartite Graph Codes

Codes	Graphs	Generator matrices
$[18,6,5]_2$ **code** $(q=2, k=6)$	 (Complete tripartite graph)	$\begin{pmatrix} 1&0&0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0 \\ 0&0&1&0&0&0&1&0&0&0&1&0&0&0&1&1&0&0 \\ 0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&0&1&1 \\ 0&0&0&0&1&0&0&0&1&0&0&0&1&0&1&0&1&0 \\ 0&0&0&0&0&1&0&0&0&1&0&0&0&1&0&1&0&1 \end{pmatrix}$
$[15,6,4]_2$ **code** $(q=3, k=6)$	 (Complete bipartite graph)	$\begin{pmatrix} 1&0&0&0&0&0&1&1&1&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&1&1&1&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&1&1&1&1 \\ 0&0&0&1&0&0&1&0&0&1&0&0&1&0&0 \\ 0&0&0&0&1&0&0&1&0&0&1&0&0&1&0 \\ 0&0&0&0&0&1&0&0&1&0&0&1&0&0&1 \end{pmatrix}$

Simplex Codes

Let k be a positive integer, $n = 2^k - 1$, and let G be a $k \times n$ matrix whose columns are all the distinct non-zero vectors of $\mathbb{F}_2^k$. Let $\mathcal{C}$ be an $[n,k,d]_2$ code that has G as its generator matrix. Then, $\mathcal{C}$ is called a binary simplex code with $d = 2^{k-1}$.

$$G = \left(\begin{array}{ccc:cccc} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{array}\right)$$

A generator matrix of $[7,3,4]_2$ simplex code

Number of symbol sets with cardinality ℓ and locality r	$r = 1$	$r = 2$	$r = 3$	ℓ-locality of $\mathcal{C}$
$\ell = 1$		n		$r_1 = 2$
$\ell = 2$			$\binom{n}{2}$	$r_2 = 3$
$\ell = 3$			$\binom{n}{3}$	$r_3 = 3$

Identity Repeated (IR) Simplex Codes

Let G be a $k \times n$ matrix constructed by adding an identity matrix in front of a generator matrix of a simplex code. Let C be an $[n, k, d]_2$ code that has G as its generator matrix. Then, C is called an IR-simplex code with $n = 2^k - 1 + k$ and $d = 2^{k-1} + 1$.

$$G = \left(\begin{array}{ccc:ccc:cccc} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{array}\right)$$

A generator matrix of $[10,3,5]_2$ IR-simplex code

Number of symbol sets with cardinality ℓ and locality r	$r = 1$	$r = 2$	$r = 3$	**ℓ-locality of C**
$\ell = 1$	$2k$	$n - 2k$		$r_1 = 2$
$\ell = 2$		$k(2n - 2k - 1)$	$\binom{n-2k}{2}$	$r_2 = 3$
$\ell = 3$		$4k(k-1)$	$\binom{n}{3} - 4k(k-1)$	$r_3 = 3$

Simplex Codes & IR-Simplex Codes

- Comparison of Localities

Number of symbol sets with cardinality ℓ and locality r		$r = 1$	$r = 2$	$r = 3$
$[9, 3, 3]_2$ repetition code	$\ell = 1$	9		
	$\boxed{\ell = 2}$	9	27	
$[7, 3, 4]_2$ simplex code	$\ell = 1$		7	
	$\ell = 2$			21
	$\boxed{\ell = 3}$			35
$[10, 3, 5]_2$ IR-simplex code	$\ell = 1$	6	4	
	$\ell = 2$		39	6
	$\ell = 3$		24	96
	$\boxed{\ell = 4}$			210

$(d-1)$ tolerance

Simplex Codes & IR-Simplex Codes

- Comparison of Localities

Number of symbol sets with cardinality ℓ and locality r		$r = 1$	$r = 2$	$r = 3$
$[9, 3, 3]_2$ repetition code	$\ell = 1$	9		
	$\ell = 2$	9	27	
$[7, 3, 4]_2$ simplex code	$\ell = 1$		$\boxed{7}$	
	$\ell = 2$			21
	$\ell = 3$			35
$[10, 3, 5]_2$ IR-simplex code	$\ell = 1$	$\boxed{6}$	$\boxed{4}$	
	$\ell = 2$		39	6
	$\ell = 3$		24	96
	$\ell = 4$			210

Average locality : 2

Average locality : 1.4

Conclusion

- introduce "generalized locality"
- improved bounds on various parameters

- complete graph codes
- complete multipartite graph codes
- generalized locality of simplex codes

- IR-simplex codes
- some good algorithm of attaching a column to G for better locality