Binary Locally Repairable Codes With Minimum Distance at Least Six Based on Partial t-Spreads

Mi-Young Nam, *Student Member, IEEE*, and Hong-Yeop Song, *Senior Member, IEEE*

***Abstract*— We propose a new construction for binary locally repairable codes (LRCs) based on a (partial) t-spread, which constructs an LRC with minimum distance $d \geq 6$ and any locality $r \geq 2$. Furthermore, we found that there are some cases in which the resulting codes are optimum in terms of the upper bound by Cadambe and Mazumdar in 2015.**

***Index Terms*— Locally repairable codes, parity-check matrix, partial t-spreads.**

I. Introduction

A DISTRIBUTED storage system (DSS) stores a huge file across the network of storage nodes. Storage nodes could become unavailable temporarily or permanently due to hardware defects or network problems. Triple replication scheme has been used in DSSs for reliability, but its higher storage overhead makes it not suitable for storing files of huge size. Erasure codes such as RS codes can reduce the storage overhead than a replication. In a DSS, the efficiency of a single node repair dominates the overall system performance since single node failures occur frequently. Classical codes such as RS codes are not optimized for an efficient repair. Therefore, investigating new codes with an efficient repair is required for a DSS.

One of the important metric that measures a repair efficiency is *locality* [8]. Locality is the number of other symbols needed to repair a symbol in a failed node [8]. Locally repairable codes (LRC) are a special class of erasure correcting codes that can repair any lost symbol by accessing small subset of other coded symbols [5], [9]–[16]. An LRC is said to have locality r if every coded symbol can be repaired by at most r other coded symbols [5]. We use the following notation:

Definition 1: An $(n, k, r)_q$ LRC is a linear code of length n, dimension k, and locality r over the finite field $\mathbb{F}_q$ with q elements. When $q = 2$ we may write it as an (n, k, r) LRC.

Gopalan *et al.* proved the upper bound of the minimum distance d of an $(n, k, r)_q$ LRC code as follows [5]:

$$d \leq n - k - \lceil \frac{k}{r} \rceil + 2. \tag{1}$$

The bound in (1) is similar to the Singleton bound. Singleton bound is not tight for codes over small finite fields. Cadambe and Mazumdar proposed the improved bound which includes the size q of the finite field as follows [6], [7]:

$$k \leq \min_{t \in Z_+} \left[tr + k_{\text{opt}}^{(q)}(n - t(r+1), d) \right], \tag{2}$$

where $k_{\text{opt}}^{(q)}(n, d)$ is the largest possible dimension of a code with length n, minimum distance d, and alphabet size q. There exist upper bounds on the value of $k_{\text{opt}}^{(q)}$. And the exact value of $k_{\text{opt}}^{(q)}$ can be found in [18] for $q = 2, 3, \ldots, 9$. We call the bound (2) the CM bound. An LRC that attains the CM bound with equality is said to be *optimum*. We also call an LRC *almost optimum* if its dimension is one less than the CM bound.

Many researchers have proposed several constructions for optimum LRCs in the sense of achieving the upper bound of (1) or (2). Codes with small alphabet size are used in practice due to simple implementation. Therefore, the optimum LRCs over small finite fields are of special interest. Tamo *et al.* proposed optimum LRCs over a finite field of size $q \geq n+1$ that attain the bound of (1) [9]. Hao *et al.* proposed optimum LRCs with $d = 3, 4$ over a finite field of size $q \geq r+2$ which attain the bound of (1) [16].

Binary LRCs (BLRCs) are of special interest since no multiplications are needed in encoding, decoding and repair. Binary LRCs meeting the CM bound (2) are constructed by using anticodes [12]. Shahabinejad *et al.* [15] proposed binary LRCs with the minimum distance 4 using a parity-check matrix with some special structure. And Hao *et al.* [14] proved that there exist only 4 classes of binary LRCs with minimum distance $d = 2$ or 4 meeting the bound (1). They also constructed binary LRCs using the parity-check matrix with the special structure as in [15].

The reliability of the code with minimum distance $d = 4$ is not enough when the code length n increases. In this letter, we propose a construction for binary LRCs with minimum distance $d \geq 6$ and any locality $r \geq 2$. The construction is based on a (partial) t-spread of vector spaces. Some of the proposed codes turned out to be optimum in terms of the CM bound (2).

The rest of the letter is organized as follows. In Section II we provide preliminaries about the maximum partial t-spread of an m-dimensional vector space over $\mathbb{F}_q$. Also, the constructions of a t-spread and a partial t-spread are reviewed. In Section III, we introduce the structure of a parity-check matrix that would be used in the construction of binary LRCs. And the construction for binary LRCs with locality $r \geq 2$ based on a (partial) t-spread is proposed and it is proved that the code has minimum distance at least 6. We also show that some codes by our construction are optimum or almost optimum in terms of the CM bound. Section IV concludes this letter.

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The authors are with the Department of Electrical and Electronic Engineering, Yonsei University, Seoul 120-749, South Korea (e-mail: hysong@yonsei.ac.kr).

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II. Vector Spaces and t-Spreads

Let q be a prime power and $\mathbb{F}_q$ be the finite field with q elements. Let $V_m(q)$ denote the vector space of dimension m over $\mathbb{F}_q$. Fix a positive integer $t < m$.

A *t-spread* of $V_m(q)$ is a collection S of t-dimensional subspaces $W_0, W_1, \ldots, W_{l-1}$ of $V_m(q)$ such that $W_i \cap W_j = \{0\}$ for $i \neq j \in \{0, 1, \ldots, l-1\}$ and $\bigcup_{i=0}^{l-1} W_i = V_m(q)$ [1]. It is easy to see that a t-spread of $V_m(q)$ exists if and only if t divides m ([2, Corollary 4.17]). From the definition, the *size* of a t-spread of $V_m(q)$ must be $\frac{q^m-1}{q^t-1}$ which is the number of t-dimensional subspaces in the t-spread [1].

A partial t-spread of $V_m(q)$ is a collection S of t-dimensional subspaces $W_0, W_1, \ldots, W_{l-1}$ of $V_m(q)$ such that $W_i \cap W_j = \{0\}$ for $i \neq j \in \{0, 1, \ldots, l-1\}$ [1]. The number l is the *size* of a partial t-spread S [1]. We call S *maximum* if it has the largest possible size [1]. Furthermore, there exist many different maximum partial t-spreads but their sizes are all the same.

Let $\mu_q(m,t)$ denote the size of a (maximum partial) t-spread of $V_m(q)$. While $\mu_q(m,t) = \frac{q^m-1}{q^t-1}$ is well known for $t \mid m$, very little is known about the exact value of $\mu_q(m,t)$ when $t \nmid m$. When $m \equiv 1 \pmod t$ it is known that $\mu_2(m,t) = \frac{2^m-2}{2^t-1} - 1$ [3].

For integers m and t such that $t \mid m$, a t-spread of a vector space $V_m(q)$ can be easily generated [4]. We will first review the construction in [4] which generates a t-spread of $V_m(q)$ when $t \mid m$. Let α be a primitive element of $\mathbb{F}_q^m$. Let $l = \frac{q^m-1}{q^t-1}$ and $\gamma = \alpha^l$. Then, we obtain a partial t-spread of $V_m(q)$ as a collection of the following subspaces U_i, where $i = 0, 1, \ldots, l-1$, where

$$U_i =< \alpha^i, \alpha^i\gamma, \ldots, \alpha^i\gamma^{t-1} > .$$

The size of a maximum partial t-spread of $V_m(q)$ is not known when $t \nmid m$. However, a construction for a partial t-spread of size $\frac{q^m - q^t(q^z-1)-1}{q^t-1}$ is presented in [17], where z is the remainder obtained when m is divided by t. The construction in [17] is an extension of the construction in [4].

Now, we will review here the construction in [17] which covers the case of $t \nmid m$. Given integers m and $t < m$, let z be the remainder obtained when m is divided by t. Define $h = t + z$. We will present the vectors in $\mathbb{F}_q^m$ as follows:

$$\mathbb{F}_q^m = \{(x, y) : x \in \mathbb{F}_q^{m-h}, y \in \mathbb{F}_q^h\}.$$

Let α and β be primitive elements of $\mathbb{F}_q^{m-h}$ and $\mathbb{F}_q^h$, respectively. Let $g = \frac{q^{m-h}-1}{q^t-1}$ and $\gamma = \alpha^g$. Then, we obtain a partial t-spread of $V_m(q)$ as a collection of the following $\mathbb{F}_q^m$-subspaces, W, U_i, and $V_{i,j}$, where $i = 0, 1, \ldots, g-1$ and $j = 0, 1, \ldots, q^h - 2$, where

$$\begin{aligned} W &= < (0, \beta^0), (0, \beta^1), \ldots, (0, \beta^{t-1}) >, \\ U_i &= < (\alpha^i, 0), (\alpha^i\gamma, 0), \ldots, (\alpha^i\gamma^{t-1}, 0) >, \end{aligned}$$

and

$$V_{i,j} =< (\alpha^i, \beta^j), (\alpha^i\gamma, \beta^{j+1}), \ldots, (\alpha^i\gamma^{t-1}, \beta^{j+t-1}) > .$$

III. Main Results

A. Parity-Check Matrices of LRCs

Let $\mathcal{C}$ be an $(n, k, r)_2$ LRC code with minimum distance d. We assume that the parity-check matrix $\mathbf{H}$ of $\mathcal{C}$ consists of two parts

$$\mathbf{H} = \begin{bmatrix} \mathbf{H}_L \\ \mathbf{H}_G \end{bmatrix}.$$

We will design the above parity check matrix such that the upper block $\mathbf{H}_L$ guarantees that the locality of the code $\mathcal{C}$ is r. For this, we fix the weight of every row in $\mathbf{H}_L$ to be less than or equal to $r + 1$. In addition, the positions for '1' in the rows in $\mathbf{H}_L$ must *cover* all n coordinates to ensure that every symbol has locality at most r.

In this letter, we will only consider the code having $\mathbf{H}_L$ that consists of pairwise disjoint $s = \frac{n}{r+1}$ row vectors of weight $r+1$. Hence, we need the condition that $r+1$ divides n. The parity-check matrix $\mathbf{H}$ can now be represented as follows:

$$\mathbf{H} = \left(\begin{array}{cccc} \overbrace{\phantom{0^{r+1}}}^{\mathcal{G}_0} & \overbrace{\phantom{0^{r+1}}}^{\mathcal{G}_1} & & \overbrace{\phantom{0^{r+1}}}^{\mathcal{G}_{s-1}} \\ \mathbf{1}^{r+1} & \mathbf{0}^{r+1} & \ldots & \mathbf{0}^{r+1} \\ \mathbf{0}^{r+1} & \mathbf{1}^{r+1} & \ldots & \mathbf{0}^{r+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}^{r+1} & \mathbf{0}^{r+1} & \ldots & \mathbf{1}^{r+1} \\ \mathbf{H}_G^0 & \mathbf{H}_G^1 & \ldots & \mathbf{H}_G^{s-1} \end{array} \right), \tag{3}$$

where $\mathbf{0}^{r+1}$ and $\mathbf{1}^{r+1}$ denote the all-zero row vector and all-one row vector of length $r+1$, respectively. We use a superscript to denote the length of a row vector. And $\mathbf{H}_G^i$ denotes the i-th $(n-k-s) \times (r+1)$ sub-matrix of $\mathbf{H}_G$.

Columns of $\mathbf{H}$ are partitioned into s groups of $r+1$ columns as shown in (3). We call the first $r+1$ columns of $\mathbf{H}$ as the group $\mathcal{G}_0$, the next $r+1$ columns as $\mathcal{G}_1$, etc. There are $s = \frac{n}{r+1}$ groups in $\mathbf{H}$.

Now, the lower sub-matrix $\mathbf{H}_G = \left[\mathbf{H}_G^0\ \mathbf{H}_G^1\ \ldots\ \mathbf{H}_G^{s-1}\right]$ will determine the minimum distance of the code $\mathcal{C}$. It is well known that the minimum distance of a linear code is at least d if and only if any $d-1$ columns of $\mathbf{H}$ are linearly independent.

Proposition 1 [16]: Consider a code $\mathcal{C}$ that is defined by the parity-check matrix $\mathbf{H}$ in (3). *If all $r+1$ columns in each $\mathbf{H}_G^i$, $i = 0, 1, \ldots, s-1$, are distinct with each other, then the code has minimum distance $d \geq 4$.*

Proof: It is enough to show that any three columns of $\mathbf{H}$ are linearly independent. For simplicity, here, we mean by 'zero' the all-zero column. It is obvious that no single column is zero. Because of the upper part, no three columns add up to zero. Now, any two columns of $\mathbf{H}$ cannot sum to zero because all the columns of $\mathbf{H}$ are distinct with each other. ■

B. Proposed Construction for Binary LRCs

In this section, we will give a construction for binary LRCs with minimum distance at least 6 and any locality $r \geq 2$. We will use the parity-check matrix as described in (3), whose columns are divided into s groups each of $r+1$ columns.

Lemma 1: Consider a code $\mathcal{C}$ that is defined by the parity-check matrix $\mathbf{H}$ in (3). *If the columns of $\mathbf{H}_G$ satisfy* *the following three conditions, then the code has minimum distance $d \geq 6$:*

(G2) *no 2 columns in a group sum to zero;*
(G41) *no 4 columns in a group sum to zero; and*
(G42) *no 4 columns, two from one group and the other two from any other group, sum to zero.*

Proof: Consider a code $\mathcal{C}$ whose parity-check matrix $\mathbf{H}$ in (3) satisfies the above three conditions for the lower part. We only need to check that any 5 columns of $\mathbf{H}$ are linearly independent. For this, we will show that no i columns of $\mathbf{H}$ sum to zero for $i = 1, 2, \ldots, 5$. For simplicity, here, we mean by 'zero' the all-zero column. The context will make it clear even if it is an abuse.

Note that (G2) implies that all the columns are distinct with each other. Therefore, similar to the proof of Proposition 1, we conclude that no two or three columns of $\mathbf{H}$ sum to zero. In fact, any odd number of columns cannot add up to zero because of the upper part. This covers the cases of 1 column, 3 columns, and 5 columns also.

The sum of any four columns, when two columns are from one group and the other two from another group, cannot sum to zero because of the condition (G42). The sum of any four columns all from the same group becomes non-zero because of the condition (G41). All other cases of four-column sum cannot be zero because of the upper part. ■

A straightforward way for the parity-check matrix $\mathbf{H}$ to satisfy the condition (G2) in Lemma 1 is to construct every sub-matrix $\mathbf{H}_G^i$, $0 \leq i \leq s-1$, using some s subspaces W_i, $i = 0, 1, \ldots, s-1$, such that $W_i \cap W_j = \{0\}$, for all $i \neq j$. We can easily obtain such a collection of s subspaces in $V_m(2)$ by finding a (partial) t-spread of $V_m(2)$ for some appropriate positive integers t and m from the given r.

Lemma 2: Consider a t-dimensional vector space V over $\mathbb{F}_2$, where $t \geq 2$. Let $B = \{b_0, b_1, \ldots, b_{t-1}\}$ be a basis of V and $C = \{c_0, c_1, \ldots, c_{\lfloor \frac{t}{2} \rfloor - 1}\}$ be a set of $\lfloor \frac{t}{2} \rfloor$ vectors such that $c_i = b_{2i} + b_{2i+1}$, for $i = 0, 1, \ldots, \lfloor \frac{t}{2} \rfloor - 1$. Then, $U = B \cup C$ satisfies the following:

- *no 2 columns in U sum to zero; and*
- *no 4 columns in U sum to zero.*

Proof: Note that the sum of two vectors of C is a sum of 4 vectors of B. Note also that the sum of two vectors from B and C is a sum of either 1 vector or 3 vectors of B. Therefore, they cannot be the all-zero vector since B is a basis. Similarly for any sum of 4 vectors from U. ■

We note that in Lemma 2 there exists some 3 vectors in U which sum to the zero vector, because, $c_i = b_{2i} + b_{2i+1}$. However, this will not be a problem in the next theorem.

Now, we will construct a binary (n, k, r) LRC $\mathcal{C}_{s,m,r}$ with any locality $r \geq 2$, whose parity check matrix is given as those in (3). In the following, we will describe how to design the lower part of $\mathbf{H}$ in (3).

Theorem 1 (Main): Given an integer $r \geq 2$, determine the smallest integer t such that

$$r + 1 \leq t + \lfloor \frac{t}{2} \rfloor. \tag{4}$$

Let $V_m(2)$ be the m-dimensional vector space over $\mathbb{F}_2$. We have to choose an integer m such that $\frac{m+1}{r} \leq l$ and that there exists a (partial) t-spread of size at least l of $V_m(2)$. Let $\{W_0, W_1, \ldots, W_{l-1}\}$ be a (partial) t-spread of size l of $V_m(2)$. Let $B_i = \{b_{i,0}, b_{i,1}, \ldots, b_{i,t-1}\}$ be a basis of $W_i \in S$ and $C_i = \{c_{i,0}, c_{i,1}, \ldots, c_{i,\lfloor \frac{t}{2} \rfloor - 1}\}$ be a set whose elements are defined as $c_{i,j} = b_{i,2j} + b_{i,2j+1}$ as in Lemma 2, for $i = 0, 1, \ldots, l-1$, and $j = 0, 1, \ldots, \lfloor \frac{t}{2} \rfloor - 1$. Finally, let $U_i = B_i \cup C_i$, for $i = 0, 1, \ldots, l-1$. Let s be an integer such that

$$\frac{m+1}{r} \leq s \leq l. \tag{5}$$

Finally, use any $r+1$ vectors in U_i to fill each sub-matrix $\mathbf{H}_G^i$ as its $r+1$ columns for $i = 0, 1, \ldots, s-1$.

Then, the code $\mathcal{C}_{s,m,r}$ has length $n = (r+1)s$, dimension $k = rs - m$, minimum distance $d \geq 6$ and locality r.

Proof: The code $\mathcal{C}_{s,m,r}$ has locality r by the shape of $\mathbf{H}$ in (3). The length and dimension come from the selected parameters easily.

verify that $\mathcal{C}_{s,m,r}$ has the minimum distance $d \geq 6$, we only need to check that any 5 columns of $\mathbf{H}$ are linearly independent. For this, it is enough to show that no i columns of $\mathbf{H}$ sum to zero for $i = 1, 2, \ldots, 5$. For this, it is enough also to show that the lower part $\mathbf{H}_G$ of $\mathbf{H}$ satisfies three conditions in Lemma 1.

Now, from Lemma 2, $\mathbf{H}_G$ satisfies (G2) and (G41) in Lemma 1. For (G42), we note that the sum of any two vectors in $\mathbf{H}_G^i$ and the sum of any two vectors in $\mathbf{H}_G^j$ are different for $i \neq j$, since $W_i \cap W_j = \{0\}$ for $i \neq j$, since $\mathbf{H}_G^i \subseteq W_i \backslash \{0\}$ for all i. ■

Remark 1: The existence of an LRC with minimum distance $d \geq 6$ and any locality $r \geq 2$ is guaranteed by the following choice of parameters. Given $r \geq 2$, let $t = \lceil \frac{2(r+1)}{3} \rceil$, and $m = 2t$. Then,

$$\frac{m+1}{r} = \frac{2t+1}{r} \leq \frac{2^m - 1}{2^t - 1} = 2^t + 1 = l,$$

and hence, from [2], there exists a t-spread of size l, and the construction in Theorem 1 applies.

Example 1: Let $\{W_0, W_1, W_2, W_3, W_4\}$ be a 2-spread of $V_4(2)$. Then, we can construct a binary $(3s, 2s-4, 2)$ LRC for $3 \leq s \leq 5$. When $s = 5$, the parity-check matrix $\mathbf{H}$ becomes

$$\mathbf{H} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix} \tag{6}$$

From this, we observe that the minimum distance is exactly 6. The remaining two codes with parameters $(3s, 2s-4, 2)$, for $s = 3, 4$, also have the minimum distance 6.

Remark 2: The code $\mathcal{C}_{s,m,r}$ has the rate

$$\frac{k}{n} = \frac{rs - m}{(r+1)s} = 1 - \frac{1 + \frac{m}{s}}{r+1}.$$

Therefore, the rate increases as $\frac{m}{s}$ decreases.

TABLE I
SOME POSSIBLE PARAMETER SELECTIONS FOR GIVEN LOCALITY r

r	t	m	l	(n, k, r)	comment
2	2	4	5	$(3s, 2s-4, 2), 3 \le s \le 5$	Ex.1, Rem.1
3	3	6	9	$(4s, 3s-6, 3), 3 \le s \le 9$	Ex.2, Rem.1
4	4	8	17	$(5s, 4s-8, 4), 3 \le s \le 17$	Rem.1
5	4	8	17	$(6s, 5s-8, 5), 2 \le s \le 17$	Rem.1
5	4	9	33	$(6s, 5s-9, 5), 2 \le s \le 33$	[3], [17]
6	5	10	33	$(7s, 6s-10, 6), 2 \le s \le 33$	Rem.1
7	6	12	65	$(8s, 7s-12, 7), 2 \le s \le 65$	Rem.1
8	6	12	65	$(9s, 8s-12, 8), 2 \le s \le 65$	Rem.1

If m becomes smaller, $k = rs - m$ will increase. However, if m increases, one may find a much bigger sized (partial) t-spread, and hence, s can be increased so that one may have much more freedom of choices for the length and the rate.

The code constructed in Theorem 1 is guaranteed to have $d \ge 6$. We note that some cases have d exactly 6. Furthermore, some of them attains the CM bound.

Corollary 1: The codes $\mathcal{C}_{s,4,2}$, for $s = 4, 5$ in Theorem 1 are optimum.

Corollary 2: The codes $\mathcal{C}_{s,5,2}$, for $s = 4, 5, \ldots, 9$ in Theorem 1 are almost optimum.

Remark 3: The code $\mathcal{C}_{3,4,2}$ in Theorem 1 can be said to be also optimum because the maximum possible dimension of a binary linear code with $n = 9$ and $d = 6$ is 2 from [18], even though it is one less than the value given by the CM bound. *This gives an example that the CM bound is not tight.*

Example 2: For locality $r = 3$, we can construct a $(4s, 3s-6, 3)$ LRC $\mathcal{C}_{s,6,3}$ for $3 \le s \le 9$ from the 3-spread $\{W_0, W_1, \ldots, W_8\}$ of $V_6(2)$. When $s = 9$, the lower part $\mathbf{H}_G$ of the parity-check matrix becomes

$$\begin{pmatrix}
101100000101011100100010101101010010 \\
001010110101001001000000100111100111 \\
001000101011010100100101000010011110 \\
011100100010101101010010010100001001 \\
010101110010001010110101001001010000 \\
000001010111001000101011010100100101
\end{pmatrix}$$

The minimum distance of this code is exactly 6. The remaining six codes for $s = 3, 4, \ldots, 8$, also have the minimum distance exactly 6.

Corollary 3: The codes $\mathcal{C}_{s,6,3}$, for $s = 3, 4, \ldots, 9$, are almost optimum in terms of the CM bound (2).

Table I shows some possible selections of parameters when the locality r is given. We denote by r, t, m, l, the locality, the dimension of subspaces that we generate, the dimension of vector space and the size of t-spread (or partial t-spread) in $V_m(q)$, respectively. Under 'comment,' Ex refers to the examples above and Rem.1 indicates that the parameters are described in Remark 1 for the t-spreads. For case $m = 9$, a partial 4-spread of size 33 constructed in either [3] or [17] is used.

IV. CONCLUDING REMARKS

In this letter, we proposed a construction for binary LRCs based on a (partial) t-spread that has minimum distance at least 6 and any locality $r \ge 2$. Some codes obtained from our construction are shown to be optimum or almost optimum in terms of the CM bound.

Remark 1 guarantees the existence of at least one LRC with a given locality $r \ge 2$. It would be interesting to check/investigate whether the parameters in Remark 1 are the best choices in some sense.

It would be interesting if an LRC with locality $r \ge 2$ and minimum distance at least 8 could be constructed in a similar method.

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