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NOTE ON A SET OF SIMULTANEOUS EQUATIONS*

(*Journal of the Indian Mathematical Society*, IV, 1912, 94—96)

1. Consider the equations

$$x_1 + x_2 + x_3 + \ldots + x_n = a_1,$$
$$x_1 y_1 + x_2 y_2 + x_3 y_3 + \ldots + x_n y_n = a_2,$$
$$x_1 y_1^2 + x_2 y_2^2 + x_3 y_3^2 + \ldots + x_n y_n^2 = a_3,$$
$$x_1 y_1^3 + x_2 y_2^3 + x_3 y_3^3 + \ldots + x_n y_n^3 = a_4,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$x_1 y_1^{2n-1} + x_2 y_2^{2n-1} + x_3 y_3^{2n-1} + \ldots + x_n y_n^{2n-1} = a_{2n},$$

where $x_1, x_2, x_3, \ldots x_n$ and $y_1, y_2, y_3, \ldots y_n$ are $2n$ unknown quantities.

Now, let us take the expression

$$\phi(\theta) \equiv \frac{x_1}{1-\theta y_1} + \frac{x_2}{1-\theta y_2} + \frac{x_3}{1-\theta y_3} + \ldots + \frac{x_n}{1-\theta y_n} \quad \ldots\ldots\ldots\ldots(1)$$

and expand it in ascending powers of θ. Then we see that the expression is equal to

$$a_1 + a_2\theta + a_3\theta^2 + \ldots + a_{2n}\theta^{2n-1} + \ldots . \quad \ldots\ldots\ldots\ldots(2)$$

But (1), when simplified, will have for its numerator an expression of the $(n-1)$th degree in θ, and for its denominator an expression of the nth degree in θ.

Thus we may suppose that

$$\phi(\theta) = \frac{A_1 + A_2\theta + A_3\theta^2 + \ldots + A_n\theta^{n-1}}{1 + B_1\theta + B_2\theta^2 + B_3\theta^3 + \ldots + B_n\theta^n} \quad \ldots\ldots\ldots\ldots(3)$$
$$= a_1 + a_2\theta + a_3\theta^2 + \ldots + a_{2n}\theta^{2n-1} + \ldots;$$

and so $\quad (1 + B_1\theta + \ldots)(a_1 + a_2\theta + \ldots) = A_1 + A_2\theta + \ldots .$

Equating the coefficients of like powers of θ, we have

$$A_1 = a_1,$$
$$A_2 = a_2 + a_1 B_1,$$
$$A_3 = a_3 + a_2 B_1 + a_1 B_2,$$
$$A_n = a_n + a_{n-1} B_1 + a_{n-2} B_2 + \ldots + a_1 B_{n-1},$$
$$0 = a_{n+1} + a_n B_1 + \ldots + a_1 B_n,$$
$$0 = a_{n+2} + a_{n+1} B_1 + \ldots + a_2 B_n,$$
$$0 = a_{n+3} + a_{n+2} B_1 + \ldots + a_3 B_n,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$0 = a_{2n} + a_{2n-1} B_1 + \ldots + a_n B_n.$$

* For a solution, by determinants, of a similar set of equations, see Burnside and Panton, *Theory of Equations*, Vol. II, p. 106, Ex. 3. [Editor, *J. Indian Math. Soc.*]

From these $B_1, B_2, \ldots B_n$ can easily be found, and since $A_1, A_2, \ldots A_n$ depend upon these values they can also be found.

Now, splitting (3) into partial fractions in the form

$$\frac{p_1}{1-q_1\theta}+\frac{p_2}{1-q_2\theta}+\frac{p_3}{1-q_3\theta}+\ldots+\frac{p_n}{1-q_n\theta},$$

and comparing with (1), we see that

$$x_1=p_1,\ y_1=q_1;$$
$$x_2=p_2,\ y_2=q_2;$$
$$x_3=p_3,\ y_3=q_3;$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

2. As an example we may solve the equations:

$$\begin{aligned}
x+y+z+u+v&=2,\\
px+qy+rz+su+tv&=3,\\
p^2x+q^2y+r^2z+s^2u+t^2v&=16,\\
p^3x+q^3y+r^3z+s^3u+t^3v&=31,\\
p^4x+q^4y+r^4z+s^4u+t^4v&=103,\\
p^5x+q^5y+r^5z+s^5u+t^5v&=235,\\
p^6x+q^6y+r^6z+s^6u+t^6v&=674,\\
p^7x+q^7y+r^7z+s^7u+t^7v&=1669,\\
p^8x+q^8y+r^8z+s^8u+t^8v&=4526,\\
p^9x+q^9y+r^9z+s^9u+t^9v&=11595,
\end{aligned}$$

where $x, y, z, u, v, p, q, r, s, t$ are the unknowns. Proceeding as before, we have

$$\frac{x}{1-\theta p}+\frac{y}{1-\theta q}+\frac{z}{1-\theta r}+\frac{u}{1-\theta s}+\frac{v}{1-\theta t}$$
$$=2+3\theta+16\theta^2+31\theta^3+103\theta^4+235\theta^5+674\theta^6+1669\theta^7+4526\theta^8+11595\theta^9+\ldots.$$

By the method of indeterminate coefficients, this can be shewn to be equal to

$$\frac{2+\theta+3\theta^2+2\theta^3+\theta^4}{1-\theta-5\theta^2+\theta^3+3\theta^4-\theta^5}.$$

Splitting this into partial fractions, we get the values of the unknowns, as follows:

$$x=-\frac{3}{5},\qquad p=-1,$$
$$y=\frac{18+\sqrt{5}}{10},\qquad q=\frac{3+\sqrt{5}}{2},$$
$$z=\frac{18-\sqrt{5}}{10},\qquad r=\frac{3-\sqrt{5}}{2},$$
$$u=-\frac{8+\sqrt{5}}{2\sqrt{5}},\qquad s=\frac{\sqrt{5}-1}{2},$$
$$v=\frac{8-\sqrt{5}}{2\sqrt{5}};\qquad t=-\frac{\sqrt{5}+1}{2}.$$

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