NOTE ON A SET OF SIMULTANEOUS EQUATIONS*

(Journal of the Indian Mathematical Society, iv, 1912, 94–96)

1. Consider the equations

\[ x_1 + x_2 + x_3 + \ldots + x_n = a_1, \]
\[ x_1y_1 + x_2y_2 + x_3y_3 + \ldots + x_ny_n = a_2, \]
\[ x_1y_1^2 + x_2y_2^2 + x_3y_3^2 + \ldots + x_ny_n^2 = a_3, \]
\[ x_1y_1^3 + x_2y_2^3 + x_3y_3^3 + \ldots + x_ny_n^3 = a_4, \]
\[ \vdots \]
\[ x_1y_1^{m-1} + x_2y_2^{m-1} + x_3y_3^{m-1} + \ldots + x_ny_n^{m-1} = a_{2n}, \]

where \( x_1, x_2, x_3, \ldots, x_n \) and \( y_1, y_2, y_3, \ldots, y_n \) are 2n unknown quantities.

Now, let us take the expression

\[ \phi (\theta) = \frac{x_1}{1 - \theta y_1} + \frac{x_2}{1 - \theta y_2} + \frac{x_3}{1 - \theta y_3} + \ldots + \frac{x_n}{1 - \theta y_n} \]

and expand it in ascending powers of \( \theta \). Then we see that the expression is equal to

\[ a_1 + a_2 \theta + a_3 \theta^2 + \ldots + a_{2n} \theta^{m-1} + \ldots \]

But (1), when simplified, will have for its numerator an expression of the 
\((n-1)\)th degree in \( \theta \), and for its denominator an expression of the \(n\)th degree in \( \theta \).

Thus we may suppose that

\[ \phi (\theta) = \frac{A_1 + A_2 \theta + A_3 \theta^2 + \ldots + A_n \theta^{n-1}}{1 + B_1 \theta + B_2 \theta^2 + B_3 \theta^3 + \ldots + B_n \theta^n} \]

and so

\[ (1 + B_1 \theta + \ldots)(a_1 + a_2 \theta + \ldots) = A_1 + A_2 \theta + \ldots \]

Equating the coefficients of like powers of \( \theta \), we have

\[ A_1 = a_1, \]
\[ A_2 = a_2 + a_1B_1, \]
\[ A_3 = a_3 + a_2B_1 + a_1B_2, \]
\[ A_n = a_n + a_{n-1}B_1 + a_{n-2}B_2 + \ldots + a_1B_{n-1}, \]
\[ 0 = a_{n+1} + a_nB_1 + \ldots + a_1B_n, \]
\[ 0 = a_{n+2} + a_{n+1}B_1 + \ldots + a_2B_n, \]
\[ 0 = a_{n+3} + a_{n+2}B_1 + \ldots + a_3B_n, \]
\[ \vdots \]
\[ 0 = a_{2n} + a_{2n-1}B_1 + \ldots + a_nB_n. \]

* For a solution, by determinants, of a similar set of equations, see Burnside and Panton, Theory of Equations, Vol. ii, p. 106, Ex. 3. [Editor, J. Indian Math. Soc.]
Note on a set of Simultaneous Equations

From these \( B_1, B_2, \ldots, B_n \) can easily be found, and since \( A_1, A_2, \ldots, A_n \) depend upon these values they can also be found.

Now, splitting (3) into partial fractions in the form

\[
\frac{p_1}{1 - q_1 \theta} + \frac{p_2}{1 - q_2 \theta} + \frac{p_3}{1 - q_3 \theta} + \ldots + \frac{p_n}{1 - q_n \theta},
\]

and comparing with (1), we see that

\[
x_1 = p_1, \quad y_1 = q_1; \\
x_2 = p_2, \quad y_2 = q_2; \\
x_3 = p_3, \quad y_3 = q_3; \\
...........................
\]

2. As an example we may solve the equations:

\[
x + y + z + u + v = 2, \\
p x + q y + r z + s u + t v = 3, \\
p^2 x + q^2 y + r^2 z + s^2 u + t^2 v = 16, \\
p^3 x + q^3 y + r^3 z + s^3 u + t^3 v = 31, \\
p^4 x + q^4 y + r^4 z + s^4 u + t^4 v = 103, \\
p^5 x + q^5 y + r^5 z + s^5 u + t^5 v = 235, \\
p^6 x + q^6 y + r^6 z + s^6 u + t^6 v = 674, \\
p^7 x + q^7 y + r^7 z + s^7 u + t^7 v = 1669, \\
p^8 x + q^8 y + r^8 z + s^8 u + t^8 v = 4526, \\
p^9 x + q^9 y + r^9 z + s^9 u + t^9 v = 11595,
\]

where \( x, y, z, u, v, p, q, r, s, t \) are the unknowns. Proceeding as before, we have

\[
\frac{x}{1 - \theta p} + \frac{y}{1 - \theta q} + \frac{z}{1 - \theta r} + \frac{u}{1 - \theta s} + \frac{v}{1 - \theta t} = 2 + 3\theta + 16\theta^2 + 31\theta^3 + 103\theta^4 + 235\theta^5 + 674\theta^6 + 1669\theta^7 + 4526\theta^8 + 11595\theta^9 + \ldots.
\]

By the method of indeterminate coefficients, this can be shown to be equal to

\[
\frac{2 + \theta + 3\theta^2 + 2\theta^3 + \theta^4}{1 - \theta - 5\theta^2 + \theta^3 + 3\theta^4 - \theta^5}.
\]

Splitting this into partial fractions, we get the values of the unknowns, as follows:

\[
x = -\frac{3}{5}, \\
y = \frac{18 + \sqrt{5}}{10}, \\
z = \frac{18 - \sqrt{5}}{10}, \\
u = -\frac{8 + \sqrt{5}}{2\sqrt{5}}, \\
v = \frac{8 - \sqrt{5}}{2\sqrt{5}};
\]

\[
p = -1, \\
q = \frac{3 + \sqrt{5}}{2}, \\
r = \frac{3 - \sqrt{5}}{2}, \\
s = \frac{\sqrt{5} - 1}{2}, \\
t = -\frac{\sqrt{5} + 1}{2}.
\]